22. Let $A = (2, 4)$, $B = (4, 5)$, $C = (6, 1)$, $T = (7, 3)$, $U = (9, 4)$, and $V = (11, 0)$. Triangles ABC and TUV are specially related to each other. Make calculations to clarify this statement, and write a few words to describe what you discover.

23. A triangle that has two sides of equal length is called *isosceles*. Make up an example of an isosceles triangle, one of whose *vertices* is $(3, 5)$. If you can, find a triangle that does not have any horizontal or vertical sides.

24. Una recently purchased two boxes of ten-inch candles — one box from a discount store, and the other from an expensive boutique. It so happens that the inexpensive candles last only three hours each, while the expensive candles last five hours each. One evening, Una hosted a dinner party and lighted two candles — one from each box — at 7:30 pm. During dessert, a guest noticed that one candle was twice as long as the other. At what time was this observation made?

25. Let $A = (1, 5)$ and $B = (3, -1)$. Verify that $P = (8, 4)$ is equidistant from A and B. Find at least two more points that are equidistant from A and B. Describe all such points.

26. Find two points on the y-axis that are 9 units from $(7, 5)$.

27. A *lattice point* is a point whose coordinates are *integers*. Find two lattice points that are exactly $\sqrt{13}$ units apart. Is it possible to find lattice points that are $\sqrt{15}$ units apart? Is it possible to form a square whose area is 18 by connecting four lattice points? Explain.

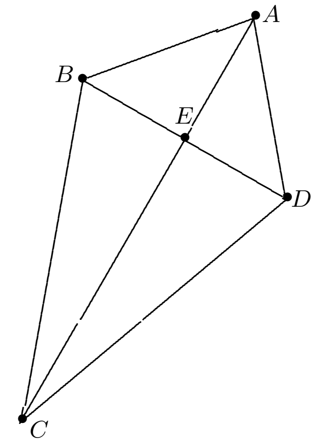
28. *Some terminology*: When two angles fit together to form a straight angle (a 180-degree angle, in other words), they are called *supplementary angles*, and either angle is the *supplement* of the other. When an angle is the same size as its supplement (a 90-degree angle), it is called a *right angle*. When two angles fit together to form a right angle, they are called complementary angles, and either angle is the *complement* of the other. Two lines that form a right angle are said to be *perpendicular*.

The three angles of a triangle fit together to form a straight angle. In one form or another, this statement is a fundamental *postulate* of Euclidean geometry — accepted as true, without proof. Taking this for granted, then, what can be said about the two non-right angles in a right triangle?

29. Let $P = (a, b)$, $Q = (0, 0)$, and $R = (-b, a)$, where a and b are positive numbers. Prove that angle PQR is right, by introducing two congruent right triangles into your diagram. Verify that the *slope* of *segment* QP is the *negative reciprocal* of the slope of segment QR.

30. Find an example of an equilateral hexagon whose sides are all $\sqrt{13}$ units long. Give lattice-point coordinates for all six points.

Here are some examples of proofs that do not use coordinates. They all show how specific *given* information can be used to logically deduce *new* information. Each example concerns a kite $ABCD$, for which $AB = AD$ and $BC = DC$ is the given information. The first two proofs show that diagonal AC creates angles BAC and DAC of the same size. The first proof consists of simple text; the second proof is written symbolically as an outline; this statement-reason form is sometimes called a "two-column proof."



Proof A: Because $AB = AD$ and $BC = DC$, and because the segment AC is shared by the triangles ABC and ADC, it follows from the SSS criterion that these triangles are congruent. Thus it is safe to conclude that the corresponding parts of these triangles are also congruent (often abbreviated to CPCTC, as in proof B below). In particular, angles BAC and DAC are the same size.

Proof B:

$AB = AD$	given
$BC = DC$	given
$AC = AC$	shared side
$\Delta ABC \cong \Delta ADC$	SSS
$\angle BAC = \angle DAC$	CPCTC

186. In the fourth line, why would writing $\Delta ABC \cong \Delta ACD$ have been incorrect?

187. Refer to the kite data above and prove that angles ABC and ADC are the same size.

Now let E be the intersection of diagonals AC and BD. The diagram makes it look like the diagonals intersect perpendicularly. Here are two proofs of this conjecture, each building on the result just proved.

Proof C: It is known that angles BAC and DAC are the same size (proof A). Because $AB = AD$ is given, and because edge AE is common to triangles BEA and DEA, it follows from the SAS criterion that these triangles are congruent. Their corresponding angles BEA and DEA must therefore be the same size. They are also supplementary, which makes them right angles, by definition.

Proof D:

$AB = AD$	given
$\angle BAE = \angle DAE$	proof B
$AE = AE$	shared side
$\Delta ABE \cong \Delta ADE$	SAS
$\angle BEA = \angle DEA$	CPCTC
$\angle BEA$ and $\angle DEA$ supplementary	E is on BD
$\angle BEA$ is right	definition of right angle

188. Using all of the above information, prove that AC bisects BD.

189. An *altitude* of a triangle is a segment that joins one of the three vertices to a point on the line that contains the opposite side, the intersection being *perpendicular*. For example, consider the triangle whose vertices are $A = (0, 0)$, $B = (8, 0)$, and $C = (4, 12)$.
(a) Find the length of the altitude from C to side AB. What is the area of ABC?
(b) Find an equation for the line that contains the altitude from A to side BC.
(c) Find an equation for the line BC.
(d) Find coordinates for the point F where the altitude from A meets side BC. It is customary to call F the *foot* of the altitude from A.
(e) Find the length of the altitude from A to side BC.
(f) As a check on your work, calculate BC and multiply it by your answer to part (e). You should be able to predict the result.
(g) It is possible to deduce the length of the altitude from B to side AC from what you have already calculated. Show how.

190. If I were to increase my cycling speed by 3 mph, I calculate that it would take me 40 seconds less time to cover each mile. What is my current cycling speed?

191. Let $A = (0, 0)$, $B = (8, 1)$, $C = (5, -5)$, $P = (0, 3)$, $Q = (7, 7)$, and $R = (1, 10)$. Prove that angles ABC and PQR have the same size.

192. (Continuation) Let D be the point on segment AB that is exactly 3 units from B, and let T be the point on segment PQ that is exactly 3 units from Q. What evidence can you give for the congruence of triangles BCD and QRT?

193. Find a point on the line $x + 2y = 8$ that is equidistant from the points $(3, 8)$ and $(9, 6)$.

194. Graph the line that is described parametrically by $(x, y) = (2t, 5 - t)$, then
(a) confirm that the point corresponding to $t = 0$ is exactly 5 units from $(3, 9)$;
(b) write a formula in terms of t for the distance from $(3, 9)$ to $(2t, 5 - t)$;
(c) find the other point on the line that is 5 units from $(3, 9)$;
(d) find the point on the line that minimizes the distance to $(3, 9)$.

195. How large a square can be put inside a right triangle whose legs are 5 cm and 12 cm?

196. You are one mile from the railroad station, and your train is due to leave in ten minutes. You have been walking at a steady rate of 3 mph, and you can run at 8 mph if you have to. For how many more minutes can you continue walking, until it becomes necessary for you to run the rest of the way to the station?

197. If a quadrilateral is equilateral, its diagonals are perpendicular. True or false? Why?

198. The diagonals AC and BD of quadrilateral $ABCD$ intersect at O. Given the information $AO = BO$ and $CO = DO$, what can you deduce about the lengths of the sides of the quadrilateral? Prove your response.

199. Let $A = (7, 7)$, $B = (5, 1)$, and $P_t = (6 + 3t, 4 - t)$. Plot A and B. Choose two values for t and plot the resulting points P_t, which should look equidistant from A and B. Make calculations to confirm the equidistance.

200. Make up a geometry problem to go with the equation $x + 3x + x\sqrt{10} = 42$.

201. Let $A = (-2, 3)$, $B = (6, 7)$, and $C = (-1, 6)$.
(a) Find an equation for the perpendicular bisector of AB.
(b) Find an equation for the perpendicular bisector of BC.
(c) Find coordinates for a point K that is equidistant from A, B, and C.

202. A segment from one of the vertices of a triangle to the midpoint of the opposite side is called a *median*. Consider the triangle defined by $A = (-2, 0)$, $B = (6, 0)$, and $C = (4, 6)$.
(a) Find an equation for the line that contains the median drawn from A to BC.
(b) Find an equation for the line that contains the median drawn from B to AC.
(c) Find coordinates for G, the intersection of the medians from A and B.
(d) Let M be the midpoint of AB. Determine whether or not M, G, and C are collinear.

203. The transformation defined by $\mathcal{T}(x, y) = (y + 2, x - 2)$ is a reflection. Verify this by calculating the effect of $\mathcal{T}$ on the triangle formed by $P = (1, 3)$, $Q = (2, 5)$, and $R = (6, 5)$. Sketch triangle PQR, find coordinates for the *image points* P', Q', and R', and sketch the *image triangle* $P'Q'R'$. Then identify the mirror line and add it to your sketch. Notice that triangle PQR is labeled in a clockwise sense; what about the labels on triangle $P'Q'R'$?

204. In quadrilateral $ABCD$, it is given that $AB = CD$ and $BC = DA$. Prove that angles ACD and CAB are the same size. N.B. If a polygon has more than three vertices, the *labeling convention* is to place the letters around the polygon in the order that they are listed. Thus AC should be one of the diagonals of $ABCD$.

205. Maintaining constant speed and direction for an hour, Whitney traveled from $(-2, 3)$ to $(10, 8)$. Where was Whitney after 35 minutes? What distance did Whitney cover in those 35 minutes?

206. A *direction vector* for a line is any vector that joins two points on that line. Find a direction vector for $2x + 5y = 8$. It is not certain that you and your classmates will get exactly the same answer. How should your answers be related, however?

207. (Continuation) Show that $[b, -a]$ is a direction vector for the line $ax + by = c$.

208. (Continuation) Show that any direction vector for the line $ax + by = c$ must be perpendicular to $[a, b]$.

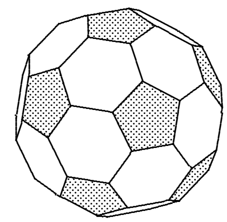
209. A particle moves according to the equation $(x, y) = (1, 2) + t[4, 3]$. Let P be the point where the path of this particle intersects the line $4x + 3y = 16$. Find coordinates for P, then explain why P is the point on $4x + 3y = 16$ that is closest to $(1, 2)$.

210. True or false? $\sqrt{4x} + \sqrt{9x} = \sqrt{13x}$

211. The line $3x+2y = 16$ is the perpendicular bisector of the segment AB. Find coordinates of point B, given that **(a)** $A = (-1, 3)$; **(b)** $A = (0, 3)$.

212. (Continuation) Point B is called the *reflection of* A *across the line* $3x + 2y = 16$; sometimes B is simply called the *image* of A. Explain this terminology. Using the same line, find another point C and its image C'. How can you check your answer?

213. A cube has 8 vertices, 12 edges, and 6 square faces. A soccer ball (also known as a *buckyball* or *truncated icosahedron*) has 12 pentagonal faces and 20 hexagonal faces. How many vertices and how many edges does a soccer ball have?



214. A rhombus has 25-cm sides, and one diagonal is 14 cm long. How long is the other diagonal?

215. Let $A = (0, 0)$ and $B = (12, 5)$, and let C be the point on segment AB that is 8 units from A. Find coordinates for C.

216. Find the point on the y-axis that is equidistant from $A = (0, 0)$ and $B = (12, 5)$.

217. Let $A = (1, 4)$, $B = (8, 0)$, and $C = (7, 8)$. Find the area of triangle ABC.

218. Sketch triangle PQR, where $P = (1, 1)$, $Q = (1, 2)$, and $R = (3, 1)$. For each of the following, apply the given transformation $\mathcal{T}$ to the vertices of triangle PQR, sketch the image triangle $P'Q'R'$, then decide which of the terms *reflection*, *rotation*, *translation*, or *glide-reflection* accurately describes the action of $\mathcal{T}$. Provide appropriate detail to justify your choices.

(a) $\mathcal{T}(x, y) = (x + 3, y - 2)$ **(b)** $\mathcal{T}(x, y) = (y, x)$

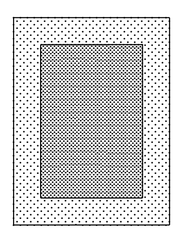
(c) $\mathcal{T}(x, y) = (-x + 2, -y + 4)$ **(d)** $\mathcal{T}(x, y) = (x + 3, -y)$

219. Prove that one of the diagonals of a kite bisects two of the angles of the kite. What about the other diagonal — must it also be an *angle bisector*? Explain your response.

220. Let $A = (2, 9)$, $B = (6, 2)$, and $C = (10, 10)$. Verify that segments AB and AC have the same length. Measure angles ABC and ACB. On the basis of your work, propose a general statement that applies to any triangle that has two sides of equal length. Prove your assertion, which might be called the *Isosceles-Triangle Theorem*.

221. If the diagonals of a quadrilateral bisect each other, then any two nonadjacent sides of the figure must have the same length. Prove that this is so.

222. Robin is mowing a rectangular field that measures 24 yards by 32 yards, by pushing the mower around and around the outside of the plot. This creates a widening border that surrounds the unmowed grass in the center. During a brief rest, Robin wonders whether the job is half done yet. How wide is the uniform mowed border when Robin *is* half done?

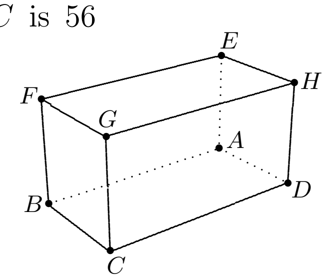


223. A geometric transformation is called an *isometry* if it preserves distances, in the following sense: The distance from M to N must be the same as the distance from M' to N', for any two points M and N and their respective images M' and N'. You have already shown in a previous exercise that any translation is an isometry. Now let $M = (a, b)$, $N = (c, d)$, $M' = (a, -b)$, and $N' = (c, -d)$. Confirm that segments MN and $M'N'$ have the same length, thereby showing that a certain transformation $\mathcal{T}$ is an isometry. What type of transformation is $\mathcal{T}$?

224. Use the distance formula to show that $\mathcal{T}(x, y) = (-y, x)$ is an isometry.

225. Triangle ABC is isosceles, with $AB = BC$, and angle BAC is 56 degrees. Find the remaining two angles of this triangle.

226. Triangle ABC is isosceles, with $AB = BC$, and angle ABC is 56 degrees. Find the remaining two angles of this triangle.



227. An ant is sitting at F, one of the vertices of a solid rectangular block. Edges AD and AE are each half the length of edge AB. The ant needs to crawl to vertex D as fast as possible. Find one of the shortest routes. How many are there?

228. Suppose that vectors $[a, b]$ and $[c, d]$ are perpendicular. Show that $ac + bd = 0$.

229. Suppose that $ac + bd = 0$. Show that vectors $[a, b]$ and $[c, d]$ are perpendicular. The number $ac + bd$ is called the *dot product* of the vectors $[a, b]$ and $[c, d]$.

230. Let $A = (0, 0)$, $B = (4, 3)$, $C = (2, 4)$, $P = (0, 4)$, and $Q = (-2, 4)$. Decide whether angles BAC and PAQ are the same size (congruent, that is), and give your reasons.

231. Let $A = (-4, 0)$, $B = (0, 6)$, and $C = (6, 0)$.
(a) Find equations for the three lines that contain the altitudes of triangle ABC.
(b) Show that the three altitudes are *concurrent*, by finding coordinates for their common point. The point of concurrence is called the *orthocenter* of triangle ABC.

232. The equation $y - 5 = m(x - 2)$ represents a line, no matter what value m has.
(a) What do all these lines have in common?
(b) When $m = -2$, what are the x- and y-intercepts of the line?
(c) When $m = -1/3$, what are the x- and y-intercepts of the line?
(d) When $m = 2$, what are the x- and y-intercepts of the line?
(e) For what values of m are the axis intercepts both positive?

233. Find the area of the triangle whose vertices are $A = (-2, 3)$, $B = (6, 7)$, and $C = (0, 6)$.

234. If triangle ABC is isosceles, with $AB = AC$, then the medians drawn from vertices B and C must have the same length. Write a *two-column proof* of this result.

235. Let $A = (-4, 0)$, $B = (0, 6)$, and $C = (6, 0)$.
(a) Find equations for the three medians of triangle ABC.
(b) Show that the three medians are concurrent, by finding coordinates for their common point. The point of concurrence is called the *centroid* of triangle ABC.

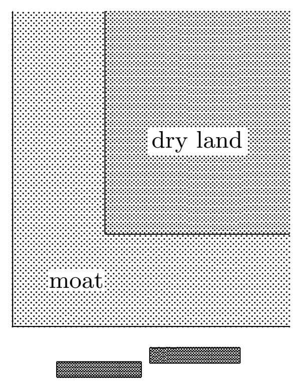
236. Given points $A = (0, 0)$ and $B = (-2, 7)$, find coordinates for points C and D so that $ABCD$ is a square.

237. Given the transformation $\mathcal{F}(x, y) = (-0.6x - 0.8y, 0.8x - 0.6y)$, Shane calculated the image of the isosceles right triangle formed by $S = (0, 0)$, $H = (0, -5)$, and $A = (5, 0)$, and declared that $\mathcal{F}$ is a reflection. Morgan instead calculated the image of the *scalene* (non-isosceles) triangle formed by $M = (7, 4)$, $O = (0, 0)$, and $R = (7, 1)$, and concluded that $\mathcal{F}$ is a rotation. Who was correct? Explain your choice, and account for the disagreement.

238. Let $A = (0, 12)$ and $B = (25, 12)$. If possible, find coordinates for a point P on the x-axis that makes angle APB a right angle.

239. Brett and Jordan are out driving in the coordinate plane, each on a separate straight road. The equations $B_t = (-3, 4) + t[1, 2]$ and $J_t = (5, 2) + t[-1, 1]$ describe their respective travels, where t is the number of minutes after noon.
(a) Make a sketch of the two roads, with arrows to indicate direction of travel.
(b) Where do the two roads intersect?
(c) How fast is Brett going? How fast is Jordan going?
(d) Do they collide? If not, who gets to the intersection first?

240. A castle is surrounded by a rectangular moat, which is of uniform width 12 feet. A corner is shown in the top view at right. The problem is to get across the moat to dry land on the other side, without using the drawbridge. To work with, you have only two rectangular planks, whose lengths are 11 feet and 11 feet, 9 inches. Show how the planks can get you across.



241. Find k so that the vectors $[4, -3]$ and $[k, -6]$
(a) point in the same direction; **(b)** are perpendicular.

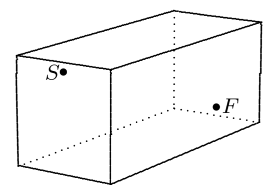
242. The lines $3x + 4y = 12$ and $3x + 4y = 72$ are parallel. Explain why. Find the distance that separates these lines. You will have to decide what "distance" means in this context.

243. Give an example of an equiangular polygon that is not equilateral.

244. Find coordinates for a point on the line $4y = 3x$ that is 8 units from $(0, 0)$.

245. An object moves with constant *velocity* (which means constant speed and direction) from $(-3, 1)$ to $(5, 7)$, taking five seconds for the trip.
(a) What is the speed of the object?
(b) Where does the object cross the y-axis?
(c) Where is the object three seconds after it leaves $(-3, 1)$?

246. A spider lived in a room that measured 30 feet long by 12 feet wide by 12 feet high. One day, the spider spied an incapacitated fly across the room, and of course wanted to crawl to it as quickly as possible. The spider was on an end wall, one foot from the ceiling and six feet from each of the long walls. The fly was stuck one foot from the floor on the opposite wall, also midway between the two long walls. Knowing some geometry, the spider cleverly took the shortest possible route to the fly and ate it for lunch. How far did the spider crawl?



247. Told to investigate the transformation $\mathcal{T}(x, y) = (x + 3, 2y + 1)$, Morgan calculated the images of $P = (1, 5)$ and $Q = (-3, 5)$. Because PQ and $P'Q'$ are equal, Morgan declared that $\mathcal{T}$ is an isometry. Shane disagreed with this conclusion. Who is correct, and why?

248. Find the area of the parallelogram whose vertices are $(0, 0)$, $(7, 2)$, $(8, 5)$, and $(1, 3)$.

249. Let $A = (0, 0)$ and $B = (12, 5)$. There are points on the y-axis that are twice as far from B as they are from A. Make a diagram that shows these points, and use it to estimate their coordinates. Then use algebra to find them exactly.

250. Given the points $A = (0, 0)$, $B = (7, 1)$, and $D = (3, 4)$, find coordinates for the point C that makes quadrilateral $ABCD$ a parallelogram. What if the question had requested $ABDC$ instead?

251. Find a vector that is perpendicular to the line $3x - 4y = 6$.

252. Measurements are made on quadrilaterals $ABCD$ and $PQRS$, and it is found that angles A, B, and C are the same size as angles P, Q, and R, respectively, and that sides AB and BC are the same length as PQ and QR, respectively. Is this enough evidence to conclude that the quadrilaterals $ABCD$ and $PQRS$ are congruent? Explain.

253. Let $P = (-1, 3)$. Find the point Q for which the line $2x + y = 5$ serves as the perpendicular bisector of segment PQ.

254. Let $A = (3, 4)$, $B = (0, -5)$, and $C = (4, -3)$. Find equations for the perpendicular bisectors of segments AB and BC, and coordinates for their common point K. Calculate lengths KA, KB, and KC. Why is K also on the perpendicular bisector of segment CA?

255. (Continuation) A circle centered at K can be drawn so that it goes through all three vertices of triangle ABC. Explain. This is why K is called the *circumcenter* of the triangle. In general, how do you locate the circumcenter of a triangle?

256. The equation $y - 5 = m(x - 2)$ represents a line, no matter what value m has.
(a) What are the x- and y-intercepts of this line?
(b) For what value of m does this line form a triangle of area 36 with the positive axes?
(c) Show that the area of a first-quadrant triangle formed by this line must be at least 20.

690. Measure the circumference and diameter of one of the 10 common circular objects supplied by your teacher. Together with your classmates, make a chart of the 10 ordered pairs (diameter, circumference).
(a) On a sheet of graph paper, make a *scatterplot* of the set of ordered pairs.
(b) Use a ruler to draw a line that fits the trend of the data. Find an equation for your line. Compare your linear equation with the results of your classmates.
(c) Give an interpretation (with units) for the slope and intercept of your line. Are there theoretical values for each of these parameters? Explain.

691. (Continuation) Not all the points in the scatterplot lie on the line you have drawn, but in some sense you have fit the line to the data. For each data point, the difference between the measured value of the circumference and the predicted value on the line at the same diameter is known as a residual. A point above the line will have a positive residual, and a point below the line will have a negative residual. Calculate the residuals for this data, and make a plot of the residuals versus the diameter for each point.

692. (Continuation) A standard method for obtaining a line of best fit for a data set is called *linear regression*, a technique that yields the line that minimizes the sum of the squares of the residuals. (This line is alternately called the *least squares line*.) Use technology to obtain an equation for the linear regression line for the diameter versus circumference data. How does this compare with the line you drew previously?

693. Let λ be the line $y = 1$ and F be the point $(-1, 2)$. Verify that the point $(2, 6)$ is equidistant from λ and F. Sketch the configuration of *all* points P that are equidistant from F and λ. Recall that this curve is called a *parabola*. Point F is called its *focus*, and line λ is called its *directrix*. Find an equation that says that $P = (x, y)$ is on the parabola.

694. (Continuation) Let $N = (2, 1)$, and find an equation for the perpendicular bisector of FN. As a check, verify that $P = (2, 6)$ is on this line. (Why could this have been predicted?) Explain why this line intersects the parabola only at P.

695. Suppose that one of the angles of a triangle is exactly twice the size of another angle of the triangle. Show that *any* such triangle can be dissected, by a single straight cut, into two triangles, one of which is isosceles, the other of which is similar to the original.

696. The line $y = x + 2$ intersects the circle $x^2 + y^2 = 10$ in two points. Call the third-quadrant point R and the first-quadrant point E, and find their coordinates. Let D be the point where the line through R and the center of the circle intersects the circle again. The chord DR is an example of a *diameter*. Show that triangle RED is a right triangle.

697. (Continuation) The portion of the circle that lies above chord RE is called an *arc*. Find a way of calculating and describing its *size*. The portion of the circle that lies below line RE is also an arc. The first arc is called a *minor* arc because it is less than half the circle, and the second arc is called a *major* arc because it is more than half the circle. It is straightforward to find the size of major arc RE once you know the size of minor arc RE. Explain how to do it.

698. For their students who turn the steering wheel too often while on the freeway, driving instructors suggest that it is better to focus on a point that is about 100 yards ahead of the car than to focus on a point only 10 yards ahead of the car. Comment on this advice.

699. Calculate the residual of $P = (1.2, 2.4)$ with respect to the line $3x + 4y = 12$.

700. Transformation $\mathcal{T}$ is defined by $\mathcal{T}(x, y) = (-5, 1) + 3[x + 5, y - 1]$. An equivalent definition is $\mathcal{T}(x, y) = (3x + 10, 3y - 2)$. Use the first definition to help you explain what kind of transformation $\mathcal{T}$ is.

701. Show that the area of a square is half the product of its diagonals. Then consider the possibility that there might be other quadrilaterals with the same property.

702. Let A and B be the positive x-intercept and the positive y-intercept, respectively, of the circle $x^2 + y^2 = 18$. Let P and Q be the positive x-intercept and the positive y-intercept, respectively, of the circle $x^2 + y^2 = 64$. Verify that the ratio of chords $AB : PQ$ matches the ratio of the corresponding diameters. What does this data suggest to you?

703. To the nearest tenth of a degree, calculate the angles of the triangle with vertices $(0, 0)$, $(6, 3)$, and $(1, 8)$.

704. In a right triangle, the 58-cm hypotenuse makes a 51-degree angle with one of the legs. To the nearest tenth of a cm, how long is that leg? Once you have the answer, find two ways to calculate the length of the other leg. They should both give the same answer.

705. Make an accurate drawing of a regular hexagon $ABCDEF$. Be prepared to report on the method you used to draw this figure. Measure the length of diagonal AC and the length of side AB. Form the *ratio* of the diagonal measurement to the side measurement. When you compare answers with your classmates, on which of these three numbers do you expect to find agreement?

706. (Continuation) Calculate $AC : AB$. One way to do it is to use trigonometry.

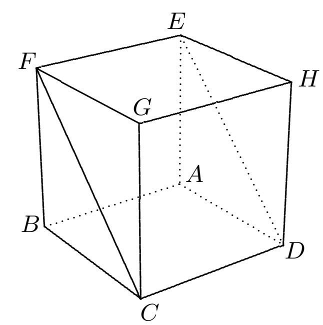
707. (Continuation) The diagonals AC, BD, CE, DF, EA, and FB form the familiar six-pointed *Star of David*. Their intersections inside $ABCDEF$ are the vertices of a smaller hexagon. Explain why the small hexagon is similar to $ABCDEF$. In particular, explain why the small hexagon is regular. Make measurements and use them to approximate the *ratio of similarity*. Then calculate an exact value for this ratio.

708. The sides of a square are parallel to the coordinate axes. Its vertices lie on the circle of radius 5 whose center is at the origin. Find coordinates for the four vertices of this square.

709. Find the lengths of both altitudes in the parallelogram determined by $[2, 3]$ and $[-5, 7]$.

710. Let $A = (0, 1)$, $B = (7, 0)$, $C = (3, 7)$, and $D = (0, 6)$. Find the areas of triangles ABC and ADC, which share side AC. Calculate the ratio of areas $ABC : ADC$. If you were to calculate the distances from B and D to the line AC, how would they compare? Explain your reasoning, or else calculate the two distances to confirm your prediction.

711. Draw a circle and label one of its diameters AB. Choose any other point on the circle and call it C. What can you say about the size of angle ACB? Does it depend on which C you chose? Justify your response.



712. The figure at right shows a cube $ABCDEFGH$. Square $ABCD$ and rectangle $EFCD$ form an angle that is called *dihedral* because it is formed by two intersecting planes. The line of intersection here is CD. Calculate the size of this angle.

713. Draw a large triangle ABC, and mark D on segment AC so that the ratio $AD : DC$ is equal to 3:4. Mark any point P on segment BD.
(a) Find the ratio of the area of triangle BAD to the area of triangle BCD.
(b) Find the ratio of the area of triangle PAD to the area of triangle PCD.
(c) Find the ratio of the area of triangle BAP to the area of triangle BCP.

714. Suppose that triangle ABC has a 30-degree angle at A and a 60-degree angle at B. Let O be the midpoint of AB. Draw the circle centered at O that goes through A. Explain why this circle also goes through B and C. Angle BOC is called a *central angle* of the circle because its vertex is at the center. The minor arc BC is called a *60-degree arc* because it *subtends* a 60-degree angle at the center. What is the *angular size* of minor arc AC? of *major* arc AC? How does the actual length of minor arc AC compare to the length of minor arc BC?

715. A triangle has two k-inch sides that make a 36-degree angle, and the third side is one inch long. Draw the bisector of one of the other angles. How long is it? There are several ways to calculate the number k. Apply at least two of them.

716. Let $A = (0, 0)$, $B = (12, 0)$, $C = (8, 6)$, and $D = (2, 6)$. The diagonals AC and BD of trapezoid $ABCD$ intersect at P. Explain why triangle ABP is similar to triangle CDP. What is the ratio of similarity? Which side of triangle CDP corresponds to side AP in triangle ABP? Why is it inaccurate to write $\triangle ABP \sim \triangle DCP$? Without finding the coordinates of P, show how you can find the lengths AP and PC.

717. (Continuation) Find the ratio of the areas of triangles
(a) ADP to CDP; **(b)** ADP to ABP; **(c)** CDP to ABP.

718. Consider the points $A = (-0.5, -8)$, $B = (0.5, -5)$, and $C = (3, 4.5)$. Calculate the residual for each of these points with respect to the line $4x - y = 7$.

719. A projector is directed towards a screen, upon which it projects a rectangular image. The following data set gives the horizontal distance of the projector lens from the screen and the resulting length of the diagonal of the rectangular image.
(a) Obtain a scatterplot of the set of ordered pairs (distance, diagonal).
(b) Find the best-fit line for the scatterplot using the least squares criterion.
(c) Give an interpretation, if appropriate, for the slope and intercept of the least squares line. Be sure to include units.

Distance from screen (cm)	236	289	336	393	438	470	506	540
Length of diagonal (cm)	103	129	152	175	195	208	226	238

720. Draw an accurate version of a regular pentagon. Be prepared to report on the method you used to draw this figure. Measure the length of a diagonal and the length of a side. Then divide the diagonal length by the side length. When you and your classmates compare answers, on which of the preceding numbers should you agree — the lengths or the ratio?

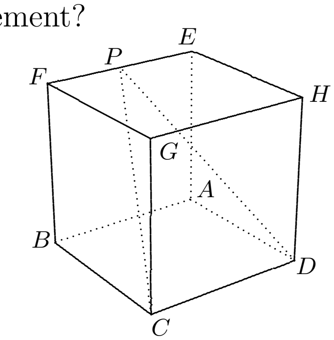
721. (Continuation) Calculate the ratio of the diagonal length to the side length in any regular pentagon. One way to do it is to use trigonometry.

722. (Continuation) Label your pentagon $ABCDE$. Draw its diagonals. They intersect to form a smaller pentagon $A'B'C'D'E'$ that lies inside $ABCDE$.
(a) Explain why $A'B'C'D'E'$ is regular, and why it is similar to $ABCDE$.
(b) Measure the length $A'B'$, and divide it by AB. Then use trigonometry to find an exact value for $A'B' : AB$, which is called the *ratio of similarity*.
(c) Consider the ways of assigning the labels A', B', C', D', and E' to the vertices of the small pentagon. How many ways are there? Is there one that stands out from the rest?

723. (Continuation) It should be possible to *circumscribe* a circle around your pentagon $ABCDE$, meaning that a circle can be drawn that goes through all five of its vertices. Find the center of this circle, and describe your method. Then measure the radius of the circle, and express your answer as a multiple of the length AB. Which of these numbers will be more useful to bring to class — the radius or the ratio?

724. If two chords of a circle have the same length, then their minor arcs have the same length too. True or false? Explain. What about the converse statement? Is it true? Why?

725. The figure at right shows a cube $ABCDEFGH$, where P the midpoint of FE.
(a) Calculate the size of the dihedral angle formed by square $ABCD$ and triangle PCD.
(b) What happens to the angle if P is a different point on FE?



726. Draw a line λ in your notebook, and mark a point F approximately an inch away from λ. Sketch the parabola that has λ as its directrix and F as its focus. Locate the point V on the parabola, called the vertex, which is *closest* to the focus. Draw the line through F that is perpendicular to λ. How is this line related to V and to the parabola?

727. Suppose that MP is a diameter of a circle centered at O, and Q is any other point on the circle. Draw the line through O that is parallel to MQ, and let R be the point where it meets minor arc PQ. Prove that R is the midpoint of minor arc PQ.

728. Line μ (*Greek* "mu") intersects segment AB at D, forming a 57-degree angle. Suppose that $AD:DB = 2:3$ is known. What can you say about the distances from A to μ and from B to μ? If $2:3$ is replaced by another ratio $m:n$, how is your answer affected?

729. The circle $x^2 + y^2 = 25$ goes through $A = (5, 0)$ and $B = (3, 4)$. To the nearest tenth of a degree, find the size of the minor arc AB.

730. An equilateral triangle is *inscribed* in the circle of radius 1 centered at the origin (the *unit circle*). If one of the vertices is $(1, 0)$, what are the coordinates of the other two? The three points divide the circle into three arcs; what are the angular sizes of these arcs?

731. On a circle whose center is O, mark points P and A so that minor arc PA is a 46-degree arc. What does this tell you about angle POA? Extend PO to meet the circle again at T. What is the size of angle PTA? This angle is *inscribed* in the circle, because all three points are on the circle. The arc PA is *intercepted* by the angle PTA.

732. (Continuation) If minor arc PA is a k-degree arc, what is the size of angle PTA?

733. The area of triangle ABC is 231 square inches, and point P is marked on side AB so that $AP:PB = 3:4$. What are the areas of triangles APC and BPC?

734. Show that the medians of any triangle divide the triangle into six smaller triangles of equal area. Are any of the small triangles necessarily congruent to each other?

735. A close look at a color television screen reveals an array of thousands of tiny red, green, and blue dots. This is because any color can be obtained as a *mixture* of these three colors. For example, if neighboring red, green, and blue dots are equally bright, the effect is white. If a blue dot is unilluminated and its red and green neighbors are equally bright, the effect is yellow. In other words, white corresponds to the red:green:blue ratio $\frac{1}{3} : \frac{1}{3} : \frac{1}{3}$ and pure yellow corresponds to $\frac{1}{2} : \frac{1}{2} : 0$. Notice that the sum of the three terms in each proportion is 1. A triangle RGB provides a simple model for this mixing of colors. The vertices represent three neighboring dots. Each point C inside the triangle represents a precise color, defined as follows: The *intensities* of the red dot, green dot, and blue dot are proportional to the *areas* of the triangles CGB, CBR, and CRG, respectively. What color is represented by the centroid of RGB? What color is represented by the midpoint of side RG?

736. (Continuation) Point C is $\frac{3}{5}$ of the way from R to G. Give a numerical description for the color mixture that corresponds to it. The color *magenta* is composed of equal intensities of red and blue, with green absent. Where is this color in the triangle?

737. (Continuation) Given that color C is defined by the red:green:blue ratio $0.4 : g : b$, where $g + b = 0.6$, what are the possible positions for C in the triangle?

738. Triangle ABC has a 53-degree angle at A, and its circumcenter is at K. Draw a good picture of this triangle, and measure the size of angle BKC. Be prepared to describe the process you used to find K. Measure the angles B and AKC of your triangle. Measure angles C and AKB. Make a conjecture about arcs intercepted by inscribed angles. Justify your assertion.

739. The area of a trapezoidal cornfield $IOWA$ is 18000 sq m. The 100-meter side IO is parallel to the 150-meter side WA. This field is divided into four sections by diagonal roads IW and OA. Find the areas of the triangular sections.

740. In triangle ABC, it is given that angle BCA is right. Let $a = BC$, $b = CA$, and $c = AB$. Using a, b, and c, express the sine, cosine, and tangent ratios of acute angles A and B.

741. The sine of a 38-degree angle is some number r. Without using a calculator, you should be able to identify the angle size whose cosine is the same number r.

742. Given SSS information about an isosceles triangle, describe the process you would use to calculate the sizes of its angles.

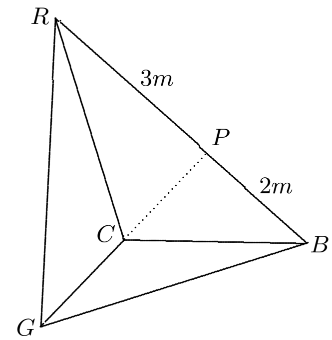
743. Draw non-parallel vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ emanating from a common point. In order that $\mathbf{u} + \mathbf{v}$ bisect the angle formed by $\mathbf{u}$ and $\mathbf{v}$, what must be true of $\mathbf{u}$ and $\mathbf{v}$?

744. If P and Q are points on a circle, then the center of the circle must be on the perpendicular bisector of chord PQ. Explain. Which point on the chord is closest to the center? Why?

745. Given that triangle ABC is similar to triangle PQR, write the three-term proportion that describes how the six sides of these figures are related.

746. Draw a circle with a 2-inch radius, mark four points randomly (not evenly spaced) on it, and label them consecutively G, E, O, and M. Measure angles GEO and GMO. Could you have predicted the result? Name another pair of angles that would have produced the same result.

747. In triangle RGB, mark P on side RB so that $RP:PB$ equals $3:2$. Let C be the midpoint of GP. Calculate the ratio of areas $CGB:CBR:CRG$. Express your answer
(a) so that the sum of the three numbers is 1;
(b) so that the three numbers are all integers.



748. Mixtures of *three* quantities can be modeled geometrically by using a triangle. What geometric figure would be suitable for describing the mixing of *two* quantities? the mixing of *four* quantities? Give the details of your models.

749. A circular park 80 meters in diameter has a straight path cutting across it. It is 24 meters from the center of the park to the closest point on this path. How long is the path?

750. Show that the lines $y = 2x - 5$ and $-2x + 11y = 25$ create chords of equal length when they intersect the circle $x^2 + y^2 = 25$. Make a large diagram, and measure the inscribed angle formed by these chords. Describe two ways of calculating its size to the nearest 0.1 degree. What is the angular size of the arc that is intercepted by this inscribed angle?

751. A triangle has a 3-inch side, a 4-inch side, and a 5-inch side. The altitude drawn to the 5-inch side cuts this side into segments of what lengths?

752. The parallel sides of a trapezoid are 8 inches and 12 inches long, while one of the non-parallel sides is 6 inches long. How far must this side be extended to meet the extension of the opposite side? What are the possible lengths for the opposite side?

753. The midline of a trapezoid is *not* concurrent with the diagonals. Explain why.

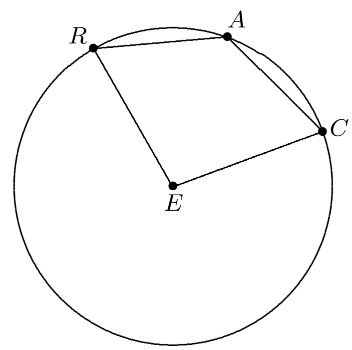
754. A chord 6 cm long is 2 cm from the center of a circle. How long is a chord that is 1 cm from the center of the same circle?

755. By using the triangle whose sides have lengths 1, $\sqrt{3}$, and 2, express the sine, cosine, and tangent of a 30-degree angle as ratios of these lengths. You can use a calculator to check your answers, of course.

756. Triangle ABC is inscribed in a circle. Given that AB is a 40-degree arc and ABC is a 50-degree angle, find the sizes of the other arcs and angles in the figure.

757. Suppose that chords AB and BC have the same lengths as chords PQ and QR, respectively, with all six points belonging to the same circle (they are *concyclic*). Is this enough information to conclude that chords AC and PR have the same length? Explain.

758. The figure at right shows points C, A, and R marked on a circle centered at E, so that chords CA and AR have the same length, and so that major arc CR is a 260-degree arc. Find the angles of quadrilateral $CARE$. What is special about the sizes of angles CAR and ACE?



759. The sides of a triangle are found to be 10 cm, 14 cm, and 16 cm long, while the sides of another triangle are found to be 15 in, 21 in, and 24 in long. On the basis of this information, what can you say about the angles of these triangles? Is it possible to *calculate* their sizes?

760. The points A, P, Q, and B appear in this order on a line, so that $AP:PQ = 2:3$ and $PQ:QB = 5:8$. Find whole numbers that are proportional to $AP:PQ:QB$.

761. A trapezoid has two 65-degree angles, and also 8-inch and 12-inch parallel sides. How long are the non-parallel sides? What is the area enclosed by this figure?

762. The data shown was generated by suspending weights, whose mass is measured in kg, from a rubber band, which stretches by the amount shown, measured in meters. Find the least squares line for this data set and interpret the results. What is the meaning of the slope in the context of this problem? How much stretch does your model predict for a 4.20 kg weight?

mass	*stretch*
0.49	0.007
0.68	0.013
0.98	0.019
1.19	0.023
1.47	0.033
1.67	0.039
1.96	0.053
2.94	0.110
3.43	0.144
3.92	0.171
4.41	0.195
4.91	0.259
5.39	0.273

763. (Continuation) The previous prediction is an example of *interpolation*, which is a prediction within the interval of values found in the data. *Extrapolation* is a prediction outside the range of the data. Is there a reason you might want to be cautious about extrapolating?

764. The dimensions of rectangle $ABCD$ are $AB = 12$ and $BC = 16$. Point P is marked on side BC, so that $BP = 5$, and the intersection of AP and BD is called T. Find the lengths of the four segments TA, TP, TB, and TD.

765. Two circles of radius 10 cm are drawn so that their centers are 12 cm apart. The two points of intersection determine a *common chord*. Find the length of this chord.

766. What is the sine of the angle whose tangent is 2? First find an answer by hand (draw a picture), then use a calculator to check.

767. Consider the line $y = 1.8x + 0.7$.
(a) Find a point whose residual with respect to this line is -1.
(b) Describe the configuration of points whose residuals are -1 with respect to this line.

768. The *median* of a set of numbers is the middle number, once the numbers have been arranged in order from least to greatest. If there are two middle numbers, then the median is half their sum. Find the median of **(a)** 5, 8, 3, 9, 5, 6, 8; **(b)** 4, 10, 8, 7.

769. True or false? The midline of a trapezoid divides the figure into two trapezoids, each similar to the original. Explain.

770. Hilary and Dale leave camp and go for a long hike. After going 7 km due east, they turn and go another 8 km in the direction 60 degrees north of east. They plan to return along a straight path. How far from camp are they at this point? Use an angle to describe the direction that Hilary and Dale should follow to reach their camp.

771. A right triangle has 6-inch, 8-inch, and 10-inch sides. A square can be inscribed in this triangle, with one vertex on each leg and two vertices on the hypotenuse. How long are the sides of the square?

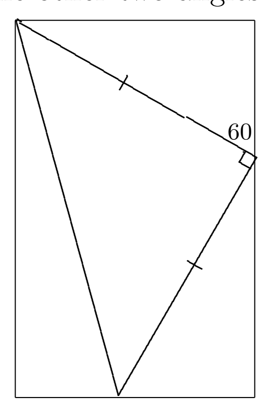
772. Find a triangle two of whose angles have sizes $\tan^{-1}(1.5)$ and $\tan^{-1}(3)$. Answer this question either by giving coordinates for the three vertices, or by giving the lengths of the three sides. To the nearest 0.1 degree, find the size of the third angle in your triangle.

773. In triangle RGB, point X divides RG according to $RX : XG = 3 : 5$, and point Y divides GB according to $GY : YB = 2 : 7$. Let C be the intersection of BX and RY.
(a) Find a ratio of whole numbers that is equal to the area ratio $CGB : CBR$.
(b) Find a ratio of whole numbers that is equal to the area ratio $CBR : CRG$.
(c) Find a ratio of whole numbers that is equal to the area ratio $CGB : CRG$.
(d) Find whole numbers m, n, and p so that $CGB : CBR : CRG = m : n : p$.
(e) The line GC cuts the side BR into two segments. What is the ratio of their lengths?

774. The area of an equilateral triangle is $100\sqrt{3}$ square inches. How long are its sides?

775. Points P, E, and A are marked on a circle whose center is R. In quadrilateral $PEAR$, angles A and E are found to be 54° and 113°, respectively. What are the other two angles?

776. The diagram shows a rectangle that has been formed by bordering an isosceles right triangle with three other right triangles, one of which has a 60-degree angle as shown. Find the sizes of the other angles in the figure. By assigning lengths to all the segments, find values for the sine, cosine, and tangent of a 75-degree angle, *without using a calculator's trigonometric functions* (except to check your formulas).



777. The points $A = (0, 13)$ and $B = (12, 5)$ lie on a circle whose center is at the origin. Write an equation for the perpendicular bisector of segment AB. Notice that this bisector goes through the origin; why was this expected?

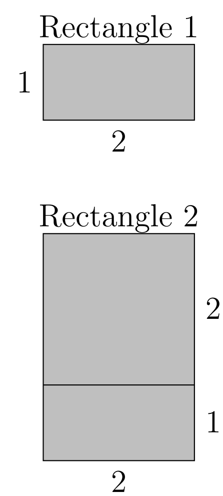
778. (Continuation) Find center and radius for another circle to which A and B both belong, and write an equation for it. How small can such a circle be? How large? What can be said about the centers of all such circles?

779. The areas of two similar triangles are 24 square cm and 54 square cm. The smaller triangle has a 6-cm side. How long is the corresponding side of the larger triangle?

780. When two circles have a common chord, their centers and the endpoints of the chord form a quadrilateral. What kind of quadrilateral is it? What special property do its diagonals have?

781. The area of triangle ABC is 75 square cm. Medians AN and CM intersect at G. What is the area of quadrilateral $GMBN$?

782. Construct a sequence of rectangles using the following procedure: The first rectangle measures 2×1 (2 units by 1 unit). Form the second rectangle by fitting a square to the longer side. (This yields a rectangle that measures 3×2.) Continue to add squares to the longer side of each successive rectangle. (The third rectangle measures 5×3.)



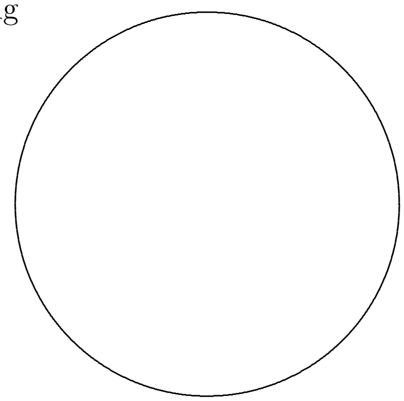
(a) Make a table of values for the width (length of the shorter side) versus length for the first five rectangles.
(b) Make a scatterplot showing width versus length and find the least squares line for the data.
(c) Examine what happens to the slope and intercept of the least squares line as you add the point for the sixth rectangle, then the seventh.
(d) The sequence of values for the widths of the rectangles is an example of a Fibonacci sequence. What does the slope of the least squares line represent?

783. Given that θ (*Greek* "theta") stands for the degree size of an acute angle, fill in the blank space between the parentheses to create a true statement: $\sin\theta = \cos(\qquad)$.

784. If corresponding sides of two similar triangles are in a $3:5$ ratio, then what is the ratio of the areas of these triangles?

785. Let $P = (-25, 0)$, $Q = (25, 0)$, and $R = (-24, 7)$.
(a) Find an equation for the circle that goes through P, Q, and R.
(b) Find at least two ways of showing that angle PRQ is right.
(c) Find coordinates for three other points R that would have made angle PRQ right.

786. Show that $(-2, 10)$, $(1, 11)$, $(6, 10)$, and $(9, 7)$ are concyclic.



787. Explain how to find the center of the circle shown, using only a pencil and a rectangular sheet of paper.

788. Trapezoid $ABCD$ has parallel sides AB and CD, of lengths 8 and 16, respectively. Diagonals AC and BD intersect at E, and the length of AC is 15. Find the lengths of AE and EC.

789. (Continuation) Through E draw the line parallel to sides AB and CD, and let P and Q be its intersections with DA and BC, respectively. Find the length of PQ.

790. Let $A = (0, 0)$, $B = (4, 0)$, and $C = (4, 3)$. Mark point D so that ACD is a right angle and DAC is a 45-degree angle. Find coordinates for D. Find the tangent of angle DAB.

791. Find a point on the line $y = x$ that lies on the parabola whose focus is $(0, 2)$ and directrix is the x-axis. Describe the relationship between the line $y = x$ and the parabola.

792. Two circles have a 24-cm common chord, their centers are 14 cm apart, and the radius of one of the circles is 13 cm. Make an accurate drawing, and find the radius for the second circle in your diagram. There are two solutions; find both.

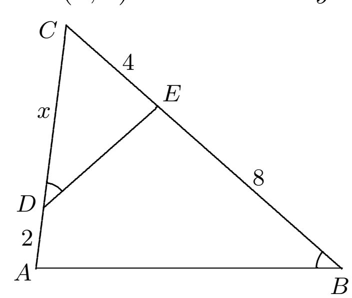
793. *SAS Similarity.* Use your protractor to carefully draw a triangle that has a 5-cm side, a 9-cm side, and whose *included angle* is 40 degrees. Construct a second triangle that has a 10-cm side, an 18-cm side, and whose included angle is also 40 degrees. Measure the remaining parts of these triangles. Could you have anticipated the results? Explain.

794. Find the perimeter of a regular 36-sided polygon *inscribed* in a circle of radius 20 cm.

795. Find the area of a regular 36-sided polygon inscribed in a circle of radius 20 cm.

796. The position of a starship is given by the equation $P_t = (18 + 3t, 24 + 4t, 110 - 5t)$. For what values of t is the starship within 100 units of a space station placed at the origin?

797. Point $P = (x, y)$ is 6 units from $A = (0, 0)$ and 9 units from $B = (9, 0)$. Find x and y.



798. Refer to the figure, in which angles ABE and CDE are equal in size, and various segments have been marked with their lengths. Find x.

799. Let $A = (0, 0)$, $B = (7, 0)$, and $C = (7, 5)$. Point D is located so that angle ACD is a right angle and the tangent of angle DAC is 5/7. Find coordinates for D.

800. A kite has an 8-inch side and a 15-inch side, which form a right angle. Find the length of the diagonals of the kite.

801. Mark points A and C on a clean sheet of paper, then spend a minute or so drawing rectangles $ABCD$. What do you notice about the configuration of points B and D?

802. What is the radius of the circumscribed circle for a triangle whose sides are 15, 15, and 24 cm long? What is the radius of the smallest circle that contains this triangle?

803. Find an equation for the circle of radius 5 whose center is at $(3, -1)$.

804. Draw a *cyclic* quadrilateral $SPAM$ in which the size of angle SPA is 110 degrees. What is the size of angle AMS? Would your answer change if M were replaced by a different point on major arc SA?

805. Let $A'B'C'$ be the midpoint triangle of triangle ABC. In other words, A', B', and C' are the midpoints of segments BC, CA, and AB, respectively. Show that triangles $A'B'C'$ and ABC have the same centroid.

806. Does $(1, 11)$ lie on the parabola defined by the focus $(0, 4)$ and the directrix $y = x$? Justify your answer.

807. Out for a walkabout in Chicago, Crocodile Dundee measures the angle of elevation to the top of the distant Willis Tower at various points along a direct path towards the building. The following data set pairs the angle of elevation in degrees with the distance walked in kilometers.
(a) Obtain the least squares line for this data set, and graph the line with a scatterplot of the data. Does the line seem to fit the data?
(b) What will happen to the angle of elevation as Dundee gets close to the building? Does the linear model accurately predict those extrapolated values?

Distance walked (km)	0.0	0.4	0.7	1.0	1.2	1.4
Angle of elevation (degrees)	3.6	3.8	4.0	4.2	4.3	4.5

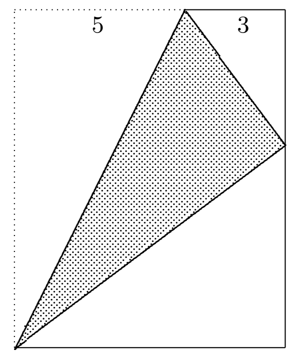
808. The area of a trapezoid is 3440 square inches, and the lengths of its parallel sides are in a 3:5 ratio. A diagonal divides the trapezoid into two triangles. What are their areas?

809. Let $WISH$ be a cyclic quadrilateral, and K be the intersection of its diagonals WS and HI. Given that arc WI is 100 degrees and arc SH is 80 degrees, find the sizes of as many angles in the figure as you can.

810. A regular dodecagon can be dissected into regular polygons (which do not all have the same number of sides). Use this dissection (but *not* a calculator) to find the area of the dodecagon, assuming that its edges are all 8 cm long.

811. Let $A = (0, 0)$ and $B = (0, 8)$. Plot several points P that make APB a 30-degree angle. Use a protractor, and be prepared to report coordinates for your points. Formulate a conjecture about the configuration of all such points.

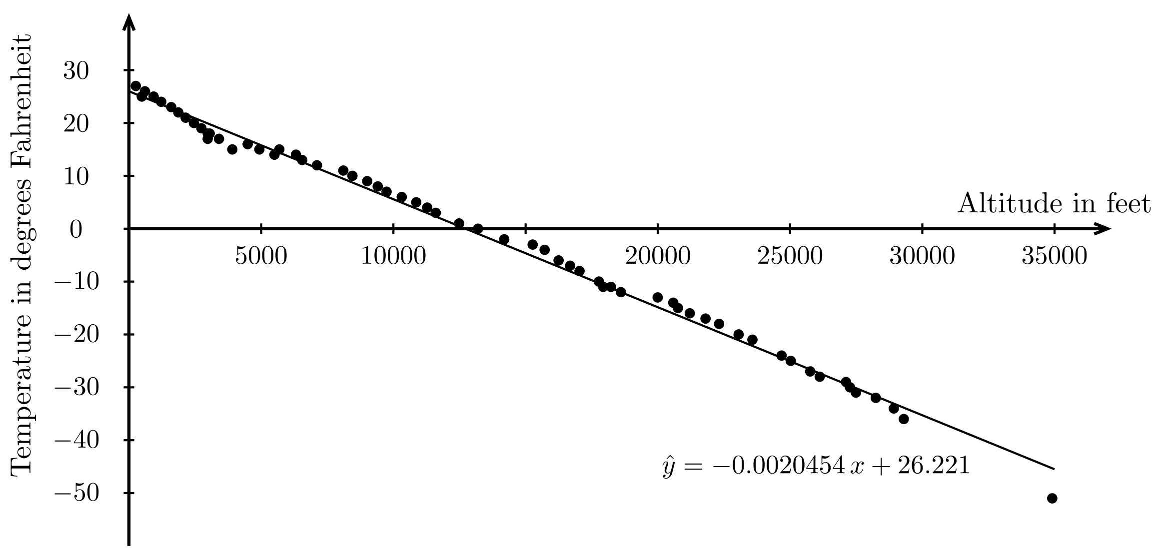
812. Triangle ABC has P on AC, Q on AB, and angle APQ equal to angle B. The lengths $AP = 3$, $AQ = 4$, and $PC = 5$ are given. Find the length of AB.



813. The figure at right shows a rectangular sheet of paper that has been creased so that one of its corners matches a point on a non-adjacent edge. Given the dimensions marked on the figure, you are to determine the length of the crease.

814. Draw the line $y = 2x - 5$ and the circle $x^2 + y^2 = 5$. Use algebra to show that these graphs touch at only one point. Find the slope of the segment that joins this point to the center of the circle, and compare your answer with the slope of the line $y = 2x - 5$. It is customary to say that a line and a circle are *tangent* if they have exactly one point in common.

815. Recently one of our math teachers was flying into Logan and decided to use the flight data on the flight monitor to record the outside temperature during the descent. The collected data is shown in the graph along with the equation for the least squares line.



(a) This data exhibits a *negative association* between temperature and altitude. What does this terminology imply about the relationship between temperature and altitude?
(b) State the slope and intercept from the regression equation. What is the meaning of each number in the context of this data? Include units in your explanation.
(c) Predict the temperature at an altitude of 30000 feet.
(d) Is it appropriate to extrapolate temperatures for altitudes outside the range of altitudes shown in the data set? Why or why not?

816. The length of segment AB is 20 cm. Find the distance from C to AB, given that C is a point on the circle that has AB as a diameter, and that
(a) $AC = CB$; **(b)** $AC = 10$ cm; **(c)** $AC = 12$ cm.

817. Quadrilateral $BAKE$ is cyclic. Extend BA to a point T outside the circle, thus producing the exterior angle KAT. Why do angles KAT and KEB have the same size?

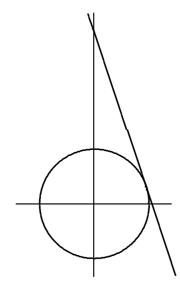
818. A kite has a 5-inch side and a 7-inch side. One of the diagonals is bisected by the other. The bisecting diagonal has length 8 inches. Find the length of the bisected diagonal.

819. Drawn in a circle whose radius is 12 cm, chord AB is 16 cm long. Calculate the angular size of minor arc AB.

820. *The reflection property of parabolas.* Consider the parabola whose focus is $F = (1, 4)$ and whose directrix is the line $x = -3$.
(a) Sketch the parabola, and make calculations that confirm that $P = (7, 12)$ is on it.
(b) Find the slope of the line μ through P that is tangent to the parabola.
(c) Calculate the size of the angle that μ makes with the line $y = 12$.
(d) Calculate the size of the angle that μ makes with segment FP. Hmm...

821. The graph of $x^2 - 6x + 9 + y^2 + 2y + 1 = 25$ is a circle. Where is the center of the circle? What is the radius of the circle?

822. Show that the line $y = 10 - 3x$ is tangent to the circle $x^2 + y^2 = 10$. Find an equation for the line perpendicular to the tangent line at the point of tangency. Show that this line goes through the center of the circle.



823. Let $K = (0, 0)$, $L = (12, 0)$, and $M = (0, 9)$. Find equations for the three lines that bisect the angles of triangle KLM. Show that the lines are concurrent at a point C, the *incenter* of KLM. Why is C called this?

824. In triangle RGB, point X divides side RG according to $RX : XG = m : n$, and point Y divides side GB according to $GY : YB = p : q$. Let C be the intersection of segments BX and RY. Find the area ratios
(a) $CGB : CBR$ **(b)** $CBR : CRG$ **(c)** $CGB : CRG$ **(d)** $CGB : CBR : CRG$
(e) Find the ratio into which the line GC divides the side BR.

825. The altitude drawn to the hypotenuse of a right triangle divides the hypotenuse into two segments, whose lengths are 8 inches and 18 inches. How long is the altitude?

826. Write an equation for the circle that is centered at $(-4, 5)$ and tangent to the x-axis.

827. Verify that the point $A = \left(8, \frac{25}{3}\right)$ lies on the parabola whose focus is $(0, 6)$ and whose directrix is the x-axis. Find an equation for the line that is tangent to the parabola at A.

828. Let $A = (1, 3)$, $B = (6, 0)$, and $C = (9, 9)$. Find the size of angle BAC. There is more than one way to do it.

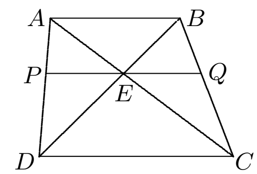
829. For (a), find center and radius. For (b), explain why it has the same graph as (a).
(a) $(x - 5)^2 + (y + 3)^2 = 49$ **(b)** $x^2 - 10x + y^2 + 6y = 15$

830. For each of the following, fill in the blank to create a perfect-square trinomial:
(a) $x^2 - 6x + \_\_\_\_$ **(b)** $y^2 + 7y + \_\_\_\_$ **(c)** $x^2 - 0.4x + \_\_\_\_$ **(d)** $y^2 - \_\_\_\_ y + 42.25$

831. Find the center and the radius of the following circles:
(a) $x^2 + y^2 - 6x + y = 3$ **(b)** $x^2 + y^2 + 8x = 0$ **(c)** $x^2 + y^2 + 2x - 8y = -8$

832. Let $K = (5, 12)$, $L = (14, 0)$, and $M = (0, 0)$. The line $x + 2y = 14$ bisects angle MLK. Find equations for the bisectors of angles KML and MKL. Is the slope of segment MK twice the slope of the bisector through M? Should it have been? Show that the three lines concur at a point C. Does C have any special significance?

833. Trapezoid $ABCD$ has parallel sides AB and CD, of lengths 12 and 18, respectively. Diagonals AC and BD intersect at E. Draw the line through E that is parallel to AB and CD, and let P and Q be its intersections with DA and BC, respectively. Find PQ.



834. The point $P = (4, 3)$ lies on the circle $x^2 + y^2 = 25$. Write an equation in standard form for the line that is tangent to the circle at P. This line meets the x-axis at a point Q. Find an equation for the other line through Q that is tangent to the circle, and identify its point of tangency. Hmm...

835. Let $P = (4, 4, 7)$, $A = (0, 0, 0)$, $B = (8, 0, 0)$, $C = (8, 8, 0)$, and $D = (0, 8, 0)$. These points are the vertices of a *regular square pyramid*. Sketch it. To the nearest tenth of a degree, find the size of the dihedral angle formed by *lateral face* PCD and *base* $ABCD$.

836. (Continuation) Find the size of the angle formed by the edge PB and the base plane $ABCD$. First you will have to decide what this means.

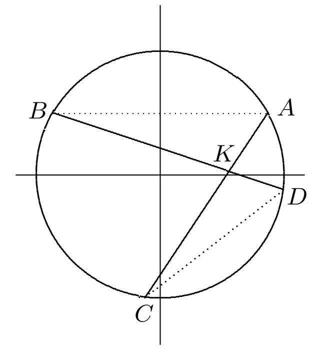
837. (Continuation) Let $Q = (5, 5, 7)$. The five points $QABCD$ are the vertices of a square pyramid. Explain why the pyramid is *not* regular. To the nearest tenth of a degree, find the size of the dihedral angle formed by the lateral face QCD and the base $ABCD$.

838. How long is the common chord of the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 4x$?

839. Draw the circles $x^2 + y^2 = 5$ and $(x - 2)^2 + (y - 6)^2 = 25$ on the same coordinate-axis system. Subtract one equation from the other, and simplify the result. This should produce a linear equation; graph it. Is there anything special about this line? Make a conjecture about what happens when one circle equation is subtracted from another.

840. Prove that the arcs between any two parallel chords in a circle must be the same size.

841. *Crossed Chords.* Verify that $A = (7, 4)$, $B = (-7, 4)$, $C = (-1, -8)$, and $D = (8, -1)$ all lie on a circle centered at the origin. Let K be the intersection of chords AC and BD. Prove that triangles KAB and KDC are similar and find the ratio of similarity. Then, show that $KA \cdot KC = KB \cdot KD$.



842. (Continuation) Explain why triangle KAD is similar to triangle KBC. What is the ratio of similarity? Is it the same ratio as for the other pair of similar triangles?

843. *Two-Tangent Theorem.* From any point P outside a given circle, there are two lines through P that are tangent to the circle. Explain why the distance from P to one of the points of tangency is the same as the distance from P to the other point of tangency. What special quadrilateral is formed by the center of the circle, the points of tangency, and P?

844. A 72-degree arc AB is drawn in a circle of radius 8 cm. How long is chord AB?

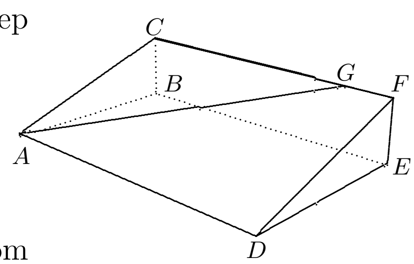
845. Find the perimeter of a regular 360-sided polygon that is inscribed in a circle of radius 5 inches. If someone did not remember the formula for the circumference of a circle, how could that person use a calculator's trigonometric functions to find the circumference of a circle with a 5-inch radius?

846. The line drawn tangent to the circle $x^2 + y^2 = 169$ at $(12, 5)$ meets the y-axis where?

847. The segments GA and GB are tangent to a circle at A and B, and AGB is a 60-degree angle. Given that $GA = 12$ cm, find the distance from G to the nearest point on the circle.

848. Through the point $(13, 0)$, there are two lines that can be drawn tangent to the circle $x^2 + y^2 = 25$. Find an equation for one of them. To begin your solution, you could find the common *length* of the tangent segments.

849. Peyton's workout today is to run repeatedly up a steep grassy slope, represented by $ADFC$ in the diagram. The workout loop is $AGCA$, in which AG requires exertion and GCA is for recovery. Point G was chosen on the ridge CF to make the slope of the climb equal 1/5. Given that $ADEB$ and $BEFC$ are rectangles, ABC is a right angle, $AD = 240$, $DE = 150$, and $EF = 50$, find the distance from point G to point C.



850. (Continuation) Peyton's next workout loop is $AHCA$, where H is a point on the path AG, chosen to make the slope of HC equal 1/5. Find the ratio AH/AG, and explain your choice.

851. A circle goes through the points A, B, C, and D consecutively. The chords AC and BD intersect at P. Given that $AP = 6$, $BP = 8$, and $CP = 3$, how long is DP?

852. Write an equation that says that $P = (x, y)$ is on the parabola whose focus is $(2, 1)$ and whose directrix is the line $y = -1$.

853. *Crossed Chords Revisited.* Suppose that A, B, D, and C lie (in that order) on a circle, and that chords AC and BD intersect, when extended, at a point P that is *outside* the circle. Explain why $PA \cdot PC = PB \cdot PD$.

854. When a regular polygon is inscribed in a circle, the circle is divided into arcs of equal size. The angular size of these arcs is simply related to the size of the interior angles of the polygon. Describe the relationship.

855. A piece of a broken circular gear is brought into a metal shop so that a replacement can be built. A ruler is placed across two points on the rim, and the length of the chord is found to be 14 inches. The distance from the midpoint of this chord to the nearest point on the rim is found to be 4 inches. Find the radius of the original gear.

856. The intersecting circles $x^2 + y^2 = 100$ and $(x - 21)^2 + y^2 = 289$ have a common chord. Find its length.

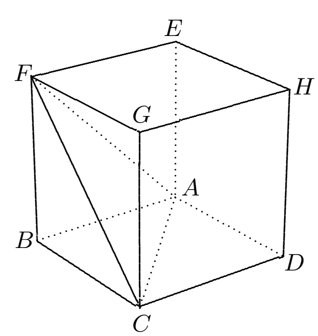
857. (Continuation) The region that is inside both circles is called a *lens*. Find the angular sizes of the two arcs that form the boundary of the lens. Does the common chord of the circles serve as a line of symmetry of the lens?

858. A triangle has two 13-cm sides and a 10-cm side. The largest circle that fits inside this triangle meets each side at a point of tangency. These points of tangency divide the sides of the triangle into segments of what lengths? What is the radius of the circle?

859. A 20-inch chord is drawn in a circle with a 12-inch radius. What is the *angular size* of the minor arc of the chord? What is the *length* of the arc, to the nearest tenth of an inch?

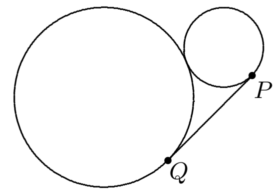
860. A triangle that has a 50-degree angle and a 60-degree angle is inscribed in a circle of radius 25 inches. The circle is divided into three arcs by the vertices of the triangle. To the nearest tenth of an inch, find the lengths of these three arcs.

861. The figure at right shows a cube $ABCDEFGH$. Triangles ABC and AFC form a dihedral angle. Notice that the line of intersection is AC. Calculate the size of this angle, to the nearest tenth of a degree.



862. What graph is traced by the parametric equation $(x, y) = (t, 4 - t^2)$?

863. A circle with a 4-inch radius is centered at A, and a circle with a 9-inch radius is centered at B, where A and B are 13 inches apart. There is a segment that is tangent to the small circle at P and to the large circle at Q. It is a common external tangent of the two circles. What kind of quadrilateral is $PABQ$? What are the lengths of its sides?

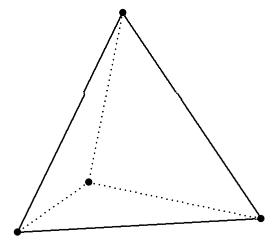


864. Segment AB, which is 25 inches long, is the diameter of a circle. Chord PQ meets AB perpendicularly at C, where $AC = 16$ in. Find the length of PQ.

865. Find the radius of the largest circle that can be drawn inside the right triangle that has 6-cm and 8-cm legs.

866. The segments GA and GB are tangent to a circle at A and B, and AGB is a 48-degree angle. Given that $GA = 12$ cm, find the distance from G to the nearest point on the circle.

867. A regular tetrahedron is a triangular pyramid, all of whose edges have the same length. If all the edges are 6-inch segments, how tall is such a pyramid, to the nearest hundredth of an inch?



868. The line $x+2y = 5$ divides the circle $x^2+y^2 = 25$ into two arcs. Calculate their lengths. The interior of the circle is divided into two regions by the line. Calculate their areas. Give three significant digits for your answers.

869. Within a given circle, is the length of a circular arc proportional to the length of its chord? Explain your answer.

870. As wildlife biologists monitor the health of animals, they would like measure the weight, which can be difficult to do with large animals in the wild. In the case of the Rocky Mountain elk, researchers decided to estimate weights by a much simpler method: measuring the chest girth. The following table gives the chest girth and weight of 19 selected elk. ["Estimating Elk Weight From Chest Girth" (Wildlife Society Bulletin [1996]: 58-61]
(a) Find the equation for the least squares line.
(b) Plot the least squares line and a scatterplot of the data. Evaluate how well the line fits the data. Is the line a reasonable model for predicting weight from girth?
(c) Predict the weight of an elk that has a chest girth of 149 cm.
(d) Interpret the intercept and slope of the least squares line, if appropriate.

Girth (cm)	96	105	108	109	110	114	122	124	132
Weight (kg)	99	196	163	196	183	171	230	225	211

Girth (cm)	135	137	138	140	142	155	157	157	159	162
Weight (kg)	231	225	266	241	264	337	284	292	300	338

871. Find an equation for the circle that goes through the points $(0, 0)$, $(0, 8)$, and $(6, 12)$. Find an equation for the line that is tangent to this circle at $(6, 12)$.

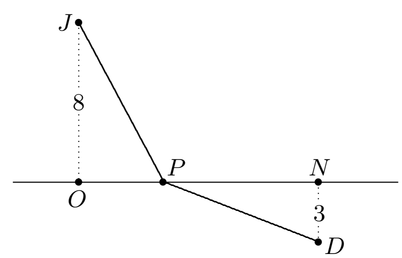
872. Can a circle always be drawn through three given points? If so, describe a procedure for finding the center of the circle. If not, explain why not.

873. A dilation $\mathcal{T}$ sends $A = (2, 3)$ to $A' = (5, 4)$, and it sends $B = (3, -1)$ to $B' = (7, -4)$. Where does it send $C = (4, 1)$? Write a general formula for $\mathcal{T}(x, y)$.

874. Find an equation for the line that goes through the two intersection points of the circle $x^2 + y^2 = 25$ and the circle $(x - 8)^2 + (y - 4)^2 = 65$.

875. All triangles and rectangles have circumscribed circles. Is this true for all kites, trapezoids, and parallelograms? Which quadrilaterals have circumscribed circles? Explain.

876. Jamie is at the point $J = (0, 8)$ offshore, needing to reach the destination $D = (12, -3)$ on land as quickly as possible. The lake shore is the x-axis. Jamie is in a boat that moves at 10 uph, with a motor bike on board that will move 20 uph once the boat reaches land. Find the landing point $P = (x, 0)$ that minimizes the total travel time from J to D. Assume that the trip from P to D is along a straight line.



877. (Continuation) Let $O = (0, 0)$ and $N = (12, 0)$. Calculate the sine of angle PJO and the sine of angle PDN. These two sine values, together with the two given speeds, fit a simple relationship: the ratio of speeds is equal to the ratio of the sines of the angles. Try to predict what you would find if the boat's speed were increased to 15 uph. To validate your prediction, re-solve the preceding problem using the new speed. Write a general statement of this principle, known as *Snell's Law*, or the *Law of Refraction*.

878. Two of the tangents to a circle meet at Q, which is 25 cm from the center. The circle has a 7-cm radius. To the nearest tenth of a degree, find the angle formed at Q by the tangents.

879. To the nearest tenth of a degree, find the angle formed by placing the vectors $[4, 3]$ and $[-7, 1]$ tail to tail.

880. Four points on a circle divide it into four arcs, whose sizes are 52 degrees, 116 degrees, 100 degrees, and 92 degrees, in consecutive order. The four points determine two intersecting chords. Find the sizes of the angles formed by the intersecting chords.

881. Let $A = (3, 4)$ and $B = (-3, 4)$, which are both on the circle $x^2 + y^2 = 25$. Let λ be the line that is tangent to the circle at A. Find the angular size of minor arc AB, then find the size of the acute angle formed by λ and chord AB. Is there a predictable relation between the two numbers? Explain.

882. What is the radius of the largest circle that will fit inside a triangle that has two 15-inch sides and an 18-inch side?

883. If a line cuts a triangle into two pieces of equal area, must that line go through the centroid of the triangle? Explain your answer.

884. One stick is 3 ft long and another is 6 ft long. You break the longer stick into sections.
(a) If the sections are 2 ft and 4 ft long, will the sticks form a triangle?
(b) If the sections are 1 ft and 5 ft long, will the sticks form a triangle?
(c) If you break the longer stick at an arbitrary point, what is the probability that they form a triangle?

885. Points D and E are marked on segments AB and BC, respectively. When segments CD and AE are drawn, they intersect at point T inside triangle ABC. It is found that segment AT is twice as long as segment TE, and that segment CT is twice as long as segment TD. Must T be the centroid of triangle ABC?

886. *An Apollonian circle.* Let $A = (-5, 0)$ and $B = (1, 0)$. Plot a few points $P = (x, y)$ for which $PA = 2PB$, including any that lie on the coordinate axes. Use the distance formula to find an equation for the configuration of all such points. Simplify your equation. Does it help you identify your graph?

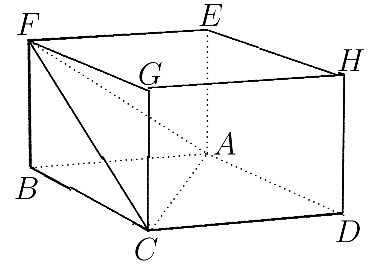
887. Sam and Kirby were out in their rowboat one day, when Kirby spied a nearby water lily. Knowing that Sam liked a mathematical challenge, Kirby announced that, with the help of the plant, it was possible to calculate the depth of the water under the boat. While Sam held the top of the plant, which remained rooted to the lake bottom during the entire process, Kirby gently rowed the boat five feet. This forced Sam's hand to the water surface. When pulled taut, the top of the plant was originally 10 inches above the water surface. Use this information to calculate the depth of the water under the boat.

888. One stick is three times as long as another. You break the longer stick at a random point. Now you have three sticks. What is the probability that they form a triangle?

889. Two sticks have length a and b with $a > b$. You break the longer stick at a random point. What is the probability that the resulting three sticks form a triangle?

890. Trapezoid $ABCD$ has parallel sides AB and CD, of lengths a and b respectively. Diagonals AC and BD intersect at E. Draw the line through E that is parallel to AB and CD, and let P and Q be its intersections with AD and BC respectively.
(a) Prove that E is the midpoint of PQ.
(b) Show that $PQ = \dfrac{1}{\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)}$. PQ is known as the *harmonic mean* of a and b.

891. You have seen that the midline of a trapezoid does *not* divide the trapezoid into two similar trapezoids. Is it possible that a *different* line (parallel to the midline) could divide the trapezoid into two similar trapezoids?



892. The figure at right shows a rectangular *prism* $ABCDEFGH$, with lengths: $CD = 8$, $DH = 5$, and $EH = 6$. Triangles ABC and AFC form a dihedral angle. To the nearest tenth of a degree, calculate the size of this dihedral angle.

893. Four points on a circle divide it into four arcs, whose sizes are 52 degrees, 116 degrees, 100 degrees, and 92 degrees, in consecutive order. When extended, the chord that belongs to the 52-degree arc intersects the chord that belongs to the 100-degree arc, at a point P outside the circle. Find the size of angle P.

894. A chord AB in a circle is extended to a point P outside the circle, and then PT is drawn tangent to the circle at T.
(a) Show that angles TAB and PTB are the same size.
(b) Show that $PT \cdot PT = PA \cdot PB$.

895. Recall that the lengths of the sides of triangle ABC are often abbreviated by writing $a = BC$, $b = CA$, and $c = AB$. Sketch triangle ABC where angle BCA is right and mark F as the foot of the perpendicular drawn from C to the hypotenuse AB. In terms of a, b, and c, express the lengths of FA, FB, and FC. The equation $c = FA + FB$ can be used to check your work.

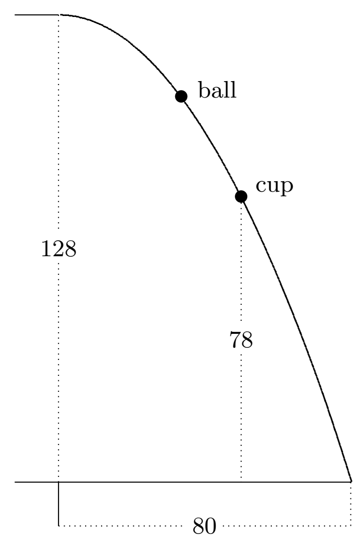
896. The Geogebra simulation found at https://www.geogebra.org/m/xfz4dfaw allows you to guess the least squares line for a given data set. The simulation allows you to change the equation of the line by dragging one of two points on the line that is shown on the graph of the data points. It also shows visually the squares (literally) of the residuals. As you move the line around the scatterplot, what is going on visually? What is the significance of the squares? How does the graph appear when you finally arrive at the least squares line?

897. The Geogebra simulation https://www.geogebra.org/m/qkxmrxdu allows the user to examine the effect of a single point on the least squares line fit to a data set for the chest girth and weight of 9 elk.
(a) The slider for a moves vertically a central point in the data set. What do you notice about the least squares line as you change the location of this point?
(b) The slider for b moves vertically a point with the lowest x-value of the data set. What do you notice about the least squares line as you change the location of this point?
(c) Compare and contrast the results for parts (a) and (b).

898. In the Assembly Hall one day, Tyler spends some time trying to figure which row gives the best view of the screen. The screen is 18 feet tall, and its bottom edge is 6 feet above eye level. Tyler finds that sitting 36 feet from the plane of the screen is not satisfactory, for the screen is far away and subtends only a 24.2-degree angle. *Verify this.* Sitting 4 feet from the screen is just as bad, because the screen subtends the same 24.2-degree angle from this position. *Verify this also.*

899. (Continuation)
(a) Intrigued by this surprising coincidence, Tyler now tries a few other distances and notices that the angles subtended by the screen at both 8 feet and 18 feet from the screen are also the same, though larger than 24.2 degrees. Find this new angle.
(b) If Tyler sits 9 feet from the screen, the angle subtended by the screen will increase, and it will equal the angle subtended when sitting k feet from the screen. Find k. See if you can determine the relationship between each of these pairs of distances that subtend equal angles.
(c) Now find the optimal viewing distance, that distance that makes the angle subtended by the screen as large as possible, and find that angle.

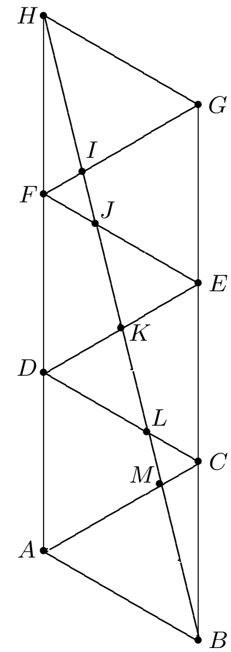
900. After rolling off the end of a ramp, a ball follows a curved trajectory to the floor. To test a theory that says that the trajectory can be described by an equation $y = h - ax^2$, Sasha makes some measurements. The end of the ramp is 128 cm above the floor, and the ball lands 80 cm downrange, as shown in the figure. In order to catch the ball in mid-flight with a cup that is 78 cm above the floor, where should Sasha place the cup?



901. It is well known that $\dfrac{a}{b} + \dfrac{c}{d}$ is *not* equivalent to $\dfrac{a+c}{b+d}$. Suppose that a, b, c, and d are all positive. Making use of the vectors $[b, a]$ and $[d, c]$, show that $\dfrac{a+c}{b+d}$ is in fact *between* the numbers $\dfrac{a}{b}$ and $\dfrac{c}{d}$, while $\dfrac{a}{b} + \dfrac{c}{d}$ is not.

902. Let $A = (0, 0)$, $B = (120, 160)$, and $C = (-75, 225)$, and let the altitudes of triangle ABC be segments AD, BE, and CF, where D, E, and F are on the sides of the triangle.
(a) Show that $(AF)(BD)(CE) = (FB)(DC)(EA)$.
(b) Show that this equation is in fact valid for *any* acute triangle ABC. (*Hint*: One way to proceed is to divide both sides of the proposed equation by $(AB)(BC)(CA)$.)

903. The figure at right is built by joining six equilateral triangles ABC, ACD, CDE, DEF, EFG, and FGH, all of whose edges are 1 unit long. It is given that $HIJKLMB$ is straight.



(a) There are five triangles in the figure that are similar to CMB. List them, making sure that you match corresponding vertices.

(b) Find the lengths of CM and EK.

(c) List the five triangles that are similar to AMB.

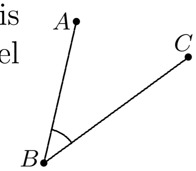
(d) Find the lengths of CL, HI, IJ, and JK.

AAA similarity: Two *triangles* are sure to be similar if their angles are equal in size.

adjacent: Two vertices of a polygon that are connected by an edge. Two edges of a polygon that intersect at a vertex. Two angles of a polygon that have a common side.

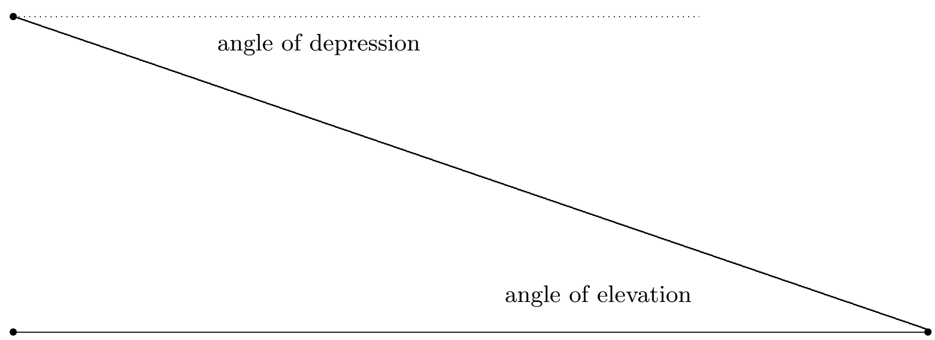
Alex in the desert: [122, 172, 357, 397, 588]

altitude: In a triangle, an altitude is a segment that joins one of the three vertices to a point on the opposite side, the intersection being perpendicular. In some triangles, it may be necessary to extend the side to meet the altitude. The *length* of this segment is also called an altitude, as is the distance that separates the parallel sides of a trapezoid. [189, 371, 436, 438]



angles can often be identified by a single letter, but sometimes three letters are necessary. The angle shown can be called B, ABC, or CBA. [8]

angle of depression: Angle formed by a horizontal ray and a line-of-sight ray that is below the horizontal. See the diagram below. [521]



angle of elevation: Angle formed by a horizontal ray and a line-of-sight ray that is above the horizontal. See the diagram above. [504]

Angle-Angle-Side (corresponding): When the parts of one triangle can be matched with the parts of another triangle, so that two pairs of corresponding angles have the same sizes, and so that one pair of corresponding sides has the same length, then the triangles are congruent. This rule of evidence is abbreviated to AAS. [120, 292]

angle bisector: Given an angle, this ray divides the angle into two equal parts. [219, 221]

Angle-Bisector Theorem: The bisector of any angle of a triangle cuts the opposite side into segments whose lengths are proportional to the adjacent sides that form the angle. [520]

Angle-Side-Angle: When the parts of one triangle can be matched with the parts of another triangle, so that two pairs of corresponding angles have the same sizes, and so that the (corresponding) shared sides have the same length, then the triangles are congruent. This rule of evidence is abbreviated to ASA. [120]

Mathematics 2 Reference

angular size of an arc: This is the size of the central angle formed by the radii that meet the endpoints of the arc. [714]

Apollonian circle: A curve consisting of those points whose distances from two fixed points are in a constant ratio. [483, 886]The Greek geometer Apollonius of Perga, who flourished about 2200 years ago, wrote many books, and gave the *parabola* its name.

arc: The portion of a circle that lies to one side of a chord is called an *arc*. [697]

arc length: The length of any arc of a circle is proportional to the size of its central angle.

areas of similar figures: If two figures are similar, then the ratio of their areas equals the *square* of the ratio of similarity. [661, 664, 779, 784]

arithmetic mean: For a set of numbers, the arithmetic mean is the sum of all the numbers in the set divided by the quantity of numbers in the set. The arithmetic mean of a and b is $\frac{a+b}{2}$. Sometimes the arithmetic mean is simply referred to as the "mean" or "average." Note that the arithmetic mean is different from the geometric mean and the harmonic mean.

bagel: [166, 506]

bisect: Divide into two pieces that are, in some sense, equal. [32, 33, 51, 201, 285]

buckyball: Named in honor of R. Buckminster Fuller, this is just another name for the *truncated icosahedron*. [213]

central angle: An angle formed by two radii of a circle. [714]

centroid: The medians of a triangle are concurrent at this point, which is the balance point (also known as the *center of gravity*) of the triangle. [235, 447, 473]

chord: A segment whose endpoints lie on a circle is called a *chord* of the circle. [683]

circle: This curve consists of all points that are at a constant distance from a *center*. The common distance is the *radius* of the circle. A segment joining the center to a point on the circle is also called a *radius*. [647, 653]

circumcenter: The perpendicular bisectors of the sides of a triangle are concurrent at this point, which is equidistant from the vertices of the triangle. [254, 255]

circumcircle: When possible, the circle that goes through all the vertices of a polygon.

collinear: Three (or more) points that all lie on a single line are *collinear*. [53, 61]

common chord: The segment that joins the points where two circles intersect. [765]

complementary: Two angles that fit together to form a right angle are called complementary. Each angle is the *complement* of the other. [28]

completing the square: Applied to an equation, this is an algebraic process that is useful for finding the center and the radius of a circle, or the vertex and focus of a parabola. [829, 830, 831]

components describe how to move from one unspecified point to another. They are obtained by *subtracting* coordinates. [74]

concentric: Two figures that have the same center are called *concentric*.

concurrent: Three (or more) lines that go through a common point are *concurrent*. [231]

concyclic: Points that all lie on a single circle are called *concyclic*. [757]

congruent: When the points of one figure can be matched with the points of another figure, so that corresponding parts have the same size, then the figures are called *congruent*, which means that they are considered to be equivalent. [21, 91, 108, 109]

converse: The converse of a statement of the form "if [something] then [something else]" is the statement "if [something else] then [something]." [366]

convex: A polygon is called *convex* if every segment joining a pair of points within it lies entirely within the polygon. [410]

coordinates: Numbers that describe the position of a point in relation to the origin of a coordinate system.

corresponding: Describes parts of figures (such as angles or segments) that have been matched by means of a transformation. [109]

cosine ratio: Given a right triangle, the cosine of one of the acute angles is the ratio of the length of the side *adjacent* to the angle to the length of the hypotenuse. The word cosine is a combination of *complement* and *sine*, so named because the cosine of an angle is the same as the sine of the complementary angle. [624, 677]

CPCTC: *Corresponding Parts of Congruent Triangles are themselves Congruent*. [page 18]

Crossed-Chords Theorem: When two chords intersect inside a circle, the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other chord. Thus the value of this product depends on only the location of the point of intersection. [841, 853]

cyclic: A polygon, all of whose vertices lie on the same circle, is called *cyclic*. Also called an *inscribed polygon*. [794, 795, 804, 817]

decagon: A polygon that has ten sides. [482]

diagonal: A segment that connects two nonadjacent vertices of a polygon.

dialation: There is no such word. See *dilation*.

diameter: A chord that goes through the center of its circle is called a *diameter*. [696]

dihedral: An angle that is formed by two intersecting *planes*. To measure its size, choose a point that is common to both planes, then through this point draw the line in each plane that is perpendicular to their line of intersection. [712, 725, 837, 861, 892]

dilation: A similarity transformation, with the special property that all lines obtained by joining points to their images are concurrent at the same *central* point. [604, 613, 619, 626]

direction vector: A vector that describes a line, by pointing from a point on the line to some other point on the line. [206]

directrix: See *parabola*.

displacement vector: The displacement vector from point (a, b) to point (c, d) is the vector $[c - a, d - b]$. [104, 113]

distance formula: The distance from (x_1, y_1) to (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, and the distance from (x_1, y_1, z_1) to (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$. These formulas are consequences of the *Pythagorean Theorem*.

dodecagon: A polygon that has twelve sides. [526]

dodecahedron: A polyhedron formed by attaching twelve polygons edge to edge. If the dodecagon is regular, each of its vertices belongs to three congruent *regular* pentagons.

Doppler shift: The change of frequency that results when the source of a signal is moving relative to the observer. [548]

dot product: Given vectors $[a, b]$ and $[m, n]$, their dot product is the number $am + bn$. Given vectors $[a, b, c]$ and $[p, q, r]$, their dot product is the number $ap + bq + cr$. In either case, it is the sum of the products of corresponding components. When the value is zero, the vectors are perpendicular, and conversely. [229, 345, 346, 347]

equiangular: A polygon all of whose angles are the same size. [13]

equidistant: A shortened form of *equally distant*. [15]

equilateral: A polygon all of whose sides have the same length. [5]

Euclidean geometry (also known as plane geometry) is characterized by its parallel postulate, which states that, *given a line, exactly one line can be drawn parallel to it through a point not on the given line.* A more familiar version of this assumption states that *the sum of the angles of a triangle is a straight angle.* [28, 342]The Greek mathematician Euclid, who flourished about 2300 years ago, wrote many books, and established a firm logical foundation for geometry.

Mathematics 2 Reference

Euler line: The centroid, the circumcenter, and the orthocenter of any triangle are collinear. [272]The Swiss scientist Leonhard Euler (1707-1783) wrote copiously on both mathematics and physics, and knew the *Aeneid* by heart.

exterior angle: An angle that is formed by a side of a polygon and the extension of an adjacent side. It is supplementary to the adjacent interior angle. [338, 349, 350, 351, 352]

Exterior-Angle Theorem: An exterior angle of a triangle is the sum of the two nonadjacent interior angles. [352, 513]

extrapolation: After having used a set of data to find a least squares line, extrapolation is the process of finding a point on the least squares line where the x-value is outside the range of the x-values of the data set. [763]

focus: See *parabola*.

foot: The point where an altitude meets the base to which it is drawn. [189, 670, 895]

function: A function is a rule that describes how an input uniquely determines an output. [10, 44, 105, 121]

geometric mean: If x and y are positive numbers, $\sqrt{xy}$ is their *geometric mean*. Note that the geometric mean is different from the arithmetic mean and the harmonic mean. [891, 899]

glide-reflection: An isometric transformation of a plane that leaves no single point fixed, but that does map a single line to itself. A glide-reflection thus maps points on either side of this line to the other side. Think of the footprints left by a person walking in a straight line. [93, 218]

Greek letters appear often in mathematics. Some of the common ones are α (alpha), β (beta), Δ or δ (delta), θ (theta), Λ and λ (lambda), μ (mu), π (pi), and Ω or ω (omega). [502, 783]

harmonic mean: The reciprocal of the arithmetic mean of the reciprocals. The *harmonic mean* of a and b is $\dfrac{1}{\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)}$. Note that the harmonic mean is different from the arithmetic mean and the geometric mean. [789, 833, 890]

head: Vector terminology for the second vertex of a directed segment. [74]

hexagon: a polygon that has six sides. [13]

Hypotenuse-Leg: When the hypotenuses of two right triangles have the same length, and a leg of one triangle has the same length as a leg of the other, then the triangles are congruent. This rule of evidence is abbreviated to HL. [286]

icosahedron: A polyhedron formed by attaching twenty polygons edge to edge. If the polyhedron is regular, each of its vertices belongs to five equilateral triangles. [213]

icosidodecahedron: A polyhedron formed by attaching the edges of twenty equilateral triangles to the edges of twelve regular pentagons. Two triangles and two pentagons meet at each vertex. [287]

image: The result of applying a transformation to a point P is called the *image point* of P and is often denoted P'. One occasionally refers to an *image segment* or an *image triangle.* [82, 83, 203]

incenter: The angle bisectors of a triangle are concurrent at this point, which is equidistant from the sides of the triangle. [823]

included angle: The angle formed by two designated segments. [793]

inscribed angle: An angle formed when two chords meet at a point on the circle. An inscribed angle is *half* the angular size of the arc it intercepts. In particular, an angle that intercepts a semicircle is a *right* angle. [731, 732]

inscribed polygon: A polygon whose vertices all lie on the same circle; also called a *cyclic polygon.* [730, 756, 794, 795]

integer: Any whole number, whether it be positive, negative, or zero. [27]

intercepted arc: The part of an arc that is found inside a given angle. [731]

interpolation: After having used a set of data to find a least squares line, interpolation is the process of finding a point on the least squares line where the x-value is inside the range of the x-values of the data set. [763]

isometry: A geometric transformation that preserves distances. The best-known examples of isometries are *translations*, *rotations*, and *reflections.* [223]

isosceles triangle: A triangle that has two sides of the same length. [23]The word is derived from the Greek *iso* + *skelos* (equal + leg)

Isosceles-Triangle Theorem: If a triangle has two sides of equal length, then the angles opposite those sides are also the same size. [220]

isosceles trapezoid: A trapezoid whose nonparallel sides have the same length. [438]

kite: A quadrilateral that has two disjoint pairs of congruent adjacent sides. [18, 187]

labeling convention: Given a polygon that has more than three vertices, place the letters around the figure in the order that they are listed. [198, 204, 250]

lateral face: Any face of a pyramid or prism that is not a base. [835]

lattice point: A point whose coordinates are both integers. [27]

lattice rectangle: A rectangle whose vertices are all lattice points. [160]

least squares line: Given a data set, the least squares line is the line through the centroid which minimizes the sum of the squares of the residuals. [692]

leg: The perpendicular sides of a right triangle are called its legs. [65]

length of a vector: This is the length of any segment that represents the vector. [74, 96]

lens: A region enclosed by two intersecting, non-concentric circular arcs. [859]

linear equation: Any straight line can be described by an equation in the form $ax+by=c$. [33, 40]

linear regression: Given a data set, a process which finds the least squares line for that set of data. [692]

lower: A term used at Phillips Exeter Academy to refer to a tenth grader. Historically the term *lower middler* has also been used. [512]

magenta: A shade of purple, named for a town in northern Italy. [736]

magnitude of a dilation: The nonnegative number obtained by dividing the length of any segment into the length of its dilated image. See *ratio of similarity*. [619, 636]

major/minor arc: Two arcs are determined by a given chord. The smaller arc is called *minor*, and the larger arc is called *major*. [697]

MasterCard: [811]

median of a triangle: A segment that joins a vertex of a triangle to the midpoint of the opposite side. [202]

midline of a trapezoid: This segment joins the midpoints of the non-parallel sides. Its length is the average of the lengths of the parallel sides, to which it is also parallel. Also known as the *median* in some books. [536]

Midline Theorem: A segment that joins the midpoints of two sides of a triangle is parallel to the third side, and is half as long. [404, 486]

midpoint: The point on a segment that is equidistant from the endpoints of the segment. If the endpoints are (a,b) and (c,d), the midpoint is $\left(\dfrac{a+c}{2}, \dfrac{b+d}{2}\right)$. [32]

mirror: See *reflection*.

negative association: A negative association between two variables is a relationship such that when one variable increases, the other decreases. [815]

negative reciprocal: One number is the negative reciprocal of another if the product of the two numbers is -1. [29]

octagon: a polygon that has eight sides. [310]

opposite: Two numbers or vectors are opposite if they differ in sign. For example, 17.5 is the opposite of -17.5, and $[2, -11]$ is the opposite of $[-2, 11]$. [96, 116]

opposite angles: In a quadrilateral, this means non-adjacent angles. [367]

opposite sides: In a quadrilateral, this means non-adjacent sides. [148]

orthocenter: The altitudes of a triangle are concurrent at this point. [231, 558]

parabola: A curve consisting of those points that are equidistant from a given line and a given point form a curve called a *parabola*. The given point is called the *focus* and the given line is called the *directrix*. The point on the parabola that is closest to the directrix (thus closest to the focus) is the *vertex*. [358, 500, 501, 502, 693, 723]

parallel: Coplanar lines that do not intersect. When drawn in a coordinate plane, they are found to have the same slope, or else no slope at all. The shorthand $\parallel$ is often used. [42]

parallelogram: A quadrilateral that has two pairs of parallel sides. [148]

parameter: See problems 46, 78, and 862.

pentagon: a polygon that has five sides. [213, 265]

perpendicular: Coplanar lines that intersect to form a right angle. If m_1 and m_2 are the slopes of two lines in the xy-plane, neither line parallel to a coordinate axis, and if $m_1 m_2 = -1$, then the lines are perpendicular. [28]

perpendicular bisector: Given a line segment, this is the line that is perpendicular to the segment and that goes through its *midpoint*. The points on this line are all *equidistant* from the endpoints of the segment. [33]

perpendicular vectors: Two vectors whose dot product is zero. [229, 584]

point-slope form: A non-vertical straight line can be described by $y - y_0 = m(x - x_0)$ or by $y = m(x - x_0) + y_0$. One of the points on the line is (x_0, y_0) and the slope is m. [42]

postulate: A statement that is accepted as true, without proof. [28]

prep: A term used at Phillips Exeter Academy to refer to a ninth grader. Historically the term *juniors* has also been used. [55]

prism: A three-dimensional figure that has two congruent and parallel *bases*, and parallelograms for its remaining *lateral faces*. If the lateral faces are all rectangles, the prism is a *right prism*. If the base is a regular polygon, the prism is also called *regular*. [311, 892]

probability: A number between 0 and 1, often expressed as a percent, that expresses the likelihood that a given event will occur. For example, the probability of rolling a 2 with a standard 6-sided die is $\frac{1}{6}$ or approximately 16.67%. [884, 888, 889]

proportion: An equation that expresses the equality of two ratios. [603, 745]

pyramid: A three-dimensional figure that is obtained by joining all the points of a polygonal *base* to a *vertex*. Thus all the lateral faces of a pyramid are triangles. If the base polygon is regular, and the lateral edges are all congruent, then the pyramid is called *regular*. [835]

Pythagorean Theorem: The square on the hypotenuse of a right triangle equals the sum of the squares on the legs. If a and b are the lengths of the legs of a right triangle, and if c is the length of the hypotenuse, then these lengths fit the Pythagorean equation $a^2 + b^2 = c^2$. [6, 16] Little is known about the Greek figure Pythagoras, who flourished about 2500 years ago, except that he probably did not discover the theorem that bears his name.

quadrant: one of the four regions formed by the coordinate axes. Quadrant I is where both coordinates are positive, and the other quadrants are numbered (using Roman numerals) in a counterclockwise fashion. [15]

quadratic formula: $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ are the two solutions to $ax^2 + bx + c = 0$.

quadrilateral: a four-sided polygon. [5, 14, 87, 100]

ray: A semi-infinite line, which starts at a point and extends infinitely far in only one direction. If a ray starts at point A, passes through point B, and the continues to infinity, we call the ray "AB".

radial expansion: See *dilation*.

ratio of similarity: The ratio of the lengths of any two corresponding segments of *similar* figures. [625]

reflection: An isometric transformation of a plane that has a line of fixed points. A reflection maps points on either side of this line (the *mirror*) to the other side. [93, 218]

reflection property of a parabola: Through any point on a *parabola*, draw the line that is parallel to the axis of symmetry, the line that goes through the focus, and the tangent line. The first two lines make equal angles with the third. [820]

regular: A polygon that is both equilateral and equiangular. [265]

regular pyramid: See *pyramid*.

residual: Given a line $y = mx + b$ and a point (x_1, y_1) not on the line, the difference $y_1 - (mx_1 + b)$ is called a *residual*. Its magnitude is the vertical distance between the point and the line. Its sign tells whether the point is above or below the line. [650]

Rhode Island School of Design.[p. 94]

rhombus: An equilateral quadrilateral. [5, 100]

right angle: An angle that is its own supplement. [28, 188]

rotation: An isometric transformation of a plane that leaves a single point fixed. [93, 218]

SAS similarity: Two triangles are certain to be similar if two sides of one triangle are proportional to two sides of the other, and if the included angles are equal in size. [793]

scalar: In the context of vectors, this is just another name for a number. [117]

scalene: A triangle no two of whose sides are the same length. [237]

scatterplot: Given a set of data, a scatter plot is a two-dimensional graph where the coordinates of the point represent the corresponding values of the two variables in the data set. [662]

segment: That part of a line that lies between two designated points. [1, 29, 32]

Sentry Theorem: The sum of the exterior angles (one at each vertex) of any polygon is 360 degrees. [352, 392, 393, 394, 513]

Shared-Altitude Theorem: If two triangles share an altitude, then the ratio of their areas is proportional to the ratio of the corresponding bases. [605]

Shared-Base Theorem: If two triangles share a base, then the ratio of their areas is proportional to the ratio of the corresponding altitudes. [634]

Side-Angle-Side: When the parts of one triangle can be matched with the parts of another triangle, so that two pairs of corresponding sides have the same lengths, and so that the (corresponding) angles they form are also the same size, then the triangles are congruent. This rule of evidence is abbreviated to just SAS. [101]

Side-Side-Angle: This is insufficient evidence for congruence. [107, 281] See the item *Hypotenuse-Leg*, however.

Side-Side-Side: When the parts of one triangle can be matched with the parts of another triangle, so that all three pairs of corresponding sides have the same lengths, then the triangles are congruent. This rule of evidence is abbreviated to just SSS. [91]

similar: Two figures are similar if their points can be matched in such a way that all ratios of corresponding lengths are proportional to a fixed *ratio of similarity.* [625]

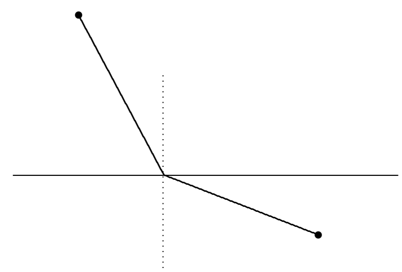
sine ratio: Given a right triangle, the sine of one of the acute angles is the ratio of the length of the side *opposite* the angle to the length of the hypotenuse. [610, 617]

skew lines: Non-intersecting lines whose direction vectors are not parallel. [669]

slope: The slope of the segment that joins the points (x_1, y_1) and (x_2, y_2) is $\dfrac{y_2 - y_1}{x_2 - x_1}$. [29, 34, 52]

slope-intercept form: Any non-vertical straight line can be described by an equation that takes the form $y = mx + b$. The slope of the line is m, and the y-intercept is b.

Snell's Law: Also known as the *Law of Refraction*, this describes the change in direction that occurs when light passes from one medium to another. The ratio of speeds is equal to the ratio of the sines of the angles formed by the rays and lines perpendicular to the interface. The Dutch physicist Willebrod Snell (1580-1626) did not tell anyone of this discovery when he made it in 1621. [876, 877]



SSS similarity: Two triangles are *similar* if their sides are proportional. [675]

stop sign: [310]

subtended angle: Given a point O and a figure $\mathcal{F}$, the *angle subtended by* $\mathcal{F}$ *at* O is the smallest angle whose vertex is O and whose interior contains $\mathcal{F}$. [714, 898, 899]

supplementary: Two angles that fit together to form a straight line are called *supplementary.* Each angle is the *supplement* of the other. [28]

symmetry axis of a parabola: The line through the focus that is perpendicular to the directrix. Except for the *vertex*, each point on the parabola is the reflected image of another point on the parabola. [726]

tail: Vector terminology for the first vertex of a directed segment. [74]

tail-to-tail: Vector terminology for directed segments with a common first vertex. [308]

tangent ratio: Given a right triangle, the tangent of one of the acute angles is the ratio of the side opposite the angle to the side adjacent to the angle. [490, 491, 492, 688]

tangent and slope: When an angle is formed by the positive x-axis and a ray through the origin, the *tangent* of the angle is the *slope* of the ray. Angles are measured in a counterclockwise sense, so that rays in the second and fourth quadrants determine negative tangent values. [490]

tangent to a circle: A line that touches a circle without crossing it. Such a line is perpendicular to the radius drawn to the point of tangency. [814]

tangent to a parabola: A line that intersects the curve without crossing it. To draw the tangent line at a given point on a parabola, join the nearest point on the directrix to the focus, then draw the perpendicular bisector of this segment. [332, 333, 334, 502, 694, 820]

tesselate: To fit non-overlapping tiles together to cover a planar region. [402, 403, 411, 489]

tetrahedron: A pyramid whose four faces are all triangles. [867]

Three-Parallels Theorem: Given three parallel lines, the segments they intercept on one transversal are proportional to the segments they intercept on any transversal. [485, 496]

transformation: A *function* that maps points to points. [83, 87, 93, 108, 169, 175, 218]

translate: To slide a figure by applying a vector to each of its points. [82, 218]

transversal: A line that intersects two other lines in a diagram. [338, 340]

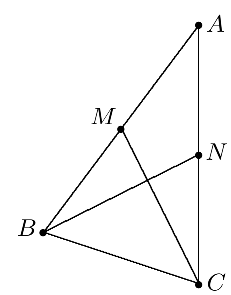
trapezoid: A quadrilateral with exactly one pair of parallel sides. If the non-parallel sides have the same length, the trapezoid is called *isosceles*. [438]

triangle inequality: For any P, Q, and R, $PQ \leq PR + RQ$. It says that any side of a triangle is less than or equal to the sum of the other two sides. [129, 456]

truncated icosahedron: A polyhedron obtained by slicing off the vertices of an icosahedron. The twelve icosahedral vertices are replaced by twelve pentagons, and the twenty icosahedral triangles become twenty hexagons. [213]

two-column proof: A way of outlining a geometric deduction. Steps are in the left column, and supporting reasons are in the right column. For example, here is how one might show that an isosceles triangle ABC has two medians of the same length. It is given that $AB = AC$ and that M and N are the midpoints of sides AB and AC, respectively. The desired conclusion is that medians CM and BN have the same length. [186]

$AB = AC$	given
$AM = AN$	M and N are midpoints
$\angle MAC = \angle NAB$	shared angle
$\Delta MAC \cong \Delta NAB$	SAS
$CM = BN$	CPCTC



Two-Tangent Theorem: From a point outside a circle, there are two segments that can be drawn tangent to the circle. These segments have the same length. [843]