Mathematics 3–4

Mathematics Department
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To the Student

Contents: Members of the PEA Mathematics Department have written the material in this book. As you work through it, you will discover that algebra, geometry, and trigonometry have been integrated into a mathematical whole. There is no Chapter 5, nor is there a section on tangents to circles. The curriculum is problem-centered, rather than topic-centered. Techniques and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records — there are no boxes containing important theorems. There is no index as such, but the reference section that starts on page 101 should help you recall the meanings of key words that are defined in the problems (where they usually appear italicized).

Problem solving: Approach each problem as an exploration. Reading each question carefully is essential, especially since definitions, highlighted in italics, are routinely inserted into the problem texts. It is important to make accurate diagrams. Here are a few useful strategies to keep in mind: create an easier problem, use the guess-and-check technique as a starting point, work backwards, recall work on a similar problem. It is important that you work on each problem when assigned, since the questions you may have about a problem will likely motivate class discussion the next day. Problem solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you're probably not the only one who is stuck, and that may even include your teacher. If you have taken the time to think about a problem, you should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.

Technology: Many of the problems in this book require the use of technology (graphing calculators, computer software, or tablet applications) in order to solve them. You are encouraged to use technology to explore, and to formulate and test conjectures. Keep the following guidelines in mind: write before you calculate, so that you will have a clear record of what you have done; be wary of rounding mid-calculation; pay attention to the degree of accuracy requested; and be prepared to explain your method to your classmates. If you don't know how to perform a needed action, there are many resources available online. Also, if you are asked to "graph $y = (2x-3)/(x+1)$", for instance, the expectation is that, although you might use a graphing tool to generate a picture of the curve, you should sketch that picture in your notebook or on the board, with correctly scaled axes.

Standardized testing: Standardized tests like the SAT, ACT, and Advanced Placement tests require calculators for certain problems, but do not allow devices with typewriter-like keyboards or internet access. For this reason, though the PEA Mathematics Department promotes the use of a variety of tools, it is still essential that students know how to use a hand-held graphing calculator to perform certain tasks. Among others, these tasks include: graphing, finding minima and maxima, creating scatter plots, regression analysis, and general numerical calculations.

Mathematics 3–4

1. From the top of Mt Washington, which is 6288 feet above sea level, how far is it to the horizon? Assume that the earth has a 3960-mile radius (one mile is 5280 feet), and give your answer to the nearest mile.

2. In mathematical discussion, a *right prism* is defined to be a solid figure that has two parallel, congruent polygonal bases, and rectangular *lateral faces.* How would you find the volume of such a figure? Explain your method.

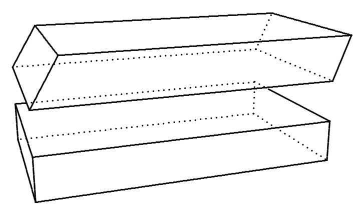
3. A chocolate company has a new candy bar in the shape of a prism whose base is a 1-inch equilateral triangle and whose sides are rectangles that measure 1 inch by 2 inches. These prisms will be packed in a box that has a regular hexagonal base with 2-inch edges, and rectangular sides that are 6 inches tall. How many candy bars fit in such a box?

4. (Continuation) The same company also markets a rectangular chocolate bar that measures 1 cm by 2 cm by 4 cm. How many of these bars can be packed in a rectangular box that measures 8 cm by 12 cm by 12 cm? How many of these bars can be packed in rectangular box that measures 8 cm by 5 cm by 5 cm? How would you pack them?

5. Starting at the same spot on a circular track that is 80 meters in diameter, Hillary and Eugene run in opposite directions, at 300 meters per minute and 240 meters per minute, respectively. They run for 50 minutes. What distance separates Hillary and Eugene when they finish? There is more than one way to interpret the word *distance* in this question.

6. Choose a positive number θ (*Greek* "theta") less than 90.0 and use a calculator to find $\sin\theta$ and $\cos\theta$. Square these numbers and add them. Could you have predicted the sum?

7. Playing cards measure 2.25 inches by 3.5 inches. A full deck of fifty-two cards is 0.75 inches high. What is the volume of a deck of cards? If the cards were uniformly shifted (turning the bottom illustration into the top illustration), would this volume be affected?



8. In the middle of the nineteenth century, octagonal barns and silos (and even some houses) became popular. How many cubic feet of grain would an octagonal silo hold if it were 12 feet tall and had a regular base with 10-foot edges?

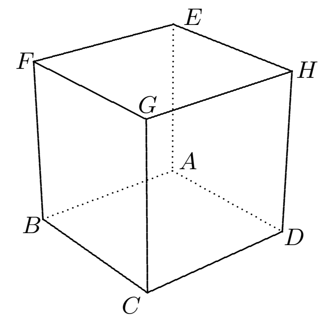
9. Build a sugar-cube pyramid as follows: First make a $5 \times 5 \times 1$ bottom layer. Then center a $4 \times 4 \times 1$ layer on the first layer, center a $3 \times 3 \times 1$ layer on the second layer, and center a $2 \times 2 \times 1$ layer on the third layer. The fifth layer is a single $1 \times 1 \times 1$ cube. Express the volume of this pyramid as a percentage of the volume of a $5 \times 5 \times 5$ cube.

10. (Continuation) Repeat the sugar-cube construction, starting with a $10 \times 10 \times 1$ base, the dimensions of each square decreasing by one unit per layer. Using a calculator, express the volume of the pyramid as a percentage of the volume of a $10 \times 10 \times 10$ cube. Repeat, using $20 \times 20 \times 1$, $50 \times 50 \times 1$, and $100 \times 100 \times 1$ bases. Do you see the trend?

11. A vector **v** of length 6 makes a 150-degree angle with the vector $[1, 0]$, when they are placed *tail-to-tail*. Find the components of **v**.

12. Why might an Earthling believe that the sun and the moon are the same size?

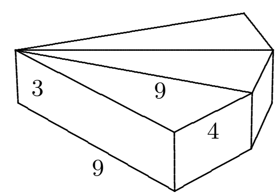
13. Given that $ABCDEFGH$ is a cube (shown at right), what is significant about the square pyramids $ADHEG$, $ABCDG$, and $ABFEG$?



14. To the nearest tenth of a degree, find the size of the angle formed by placing the vectors $[4, 0]$ and $[-6, 5]$ tail-to-tail at the origin. It is understood in questions such as this that the answer is smaller than 180 degrees.

15. Flying at an altitude of 39 000 feet one clear day, Cameron looked out the window of the airplane and wondered how far it was to the horizon. Rounding your answer to the nearest mile, answer Cameron's question.

16. A triangular prism of cheese is measured and found to be 3 inches tall. The edges of its base are 9, 9, and 4 inches long. Several congruent prisms are to be arranged around a common 3-inch segment, as shown. How many prisms can be accommodated? To the nearest cubic inch, what is their total volume?



17. The Great Pyramid at Gizeh was originally 483 feet tall, and it had a square base that was 756 feet on a side. It was built from rectangular stone blocks measuring 7 feet by 7 feet by 14 feet. Such a block weighs seventy tons. Approximately how many tons of stone were used to build the Great Pyramid? The volume of a pyramid is one third the base area times the height.

18. The angle formed by placing the vectors $[4, 0]$ and $[a, b]$ tail-to-tail at the origin is 124 degrees. The length of $[a, b]$ is 12. Find a and b.

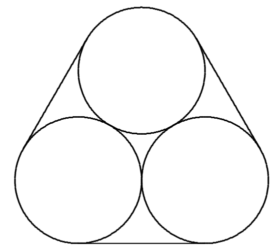
19. Pyramid $TABCD$ has a 20-cm square base $ABCD$. The edges that meet at T are 27 cm long. Make a diagram of $TABCD$, showing F, the point of $ABCD$ closest to T. Find the height TF, to the nearest cm. Find the volume of $TABCD$, to the nearest cc.

20. (Continuation) Let P be a point on edge AB, and consider the possible sizes of angle TPF. What position for P makes this angle as small as it can be? How do you know?

21. (Continuation) Let K, L, M, and N be the points on TA, TB, TC, and TD, respectively, that are 18 cm from T. What can be said about polygon $KLMN$? Explain.

22. A wheel of radius one foot is placed so that its center is at the origin, and a paint spot on the rim is at $(1, 0)$. The wheel is spun 27 degrees in a counterclockwise direction. What are the coordinates of the paint spot? What if the wheel is spun θ degrees instead?

23. The figure shows three circular pipes, all with 12-inch diameters, that are strapped together by a metal band. How long is the band?



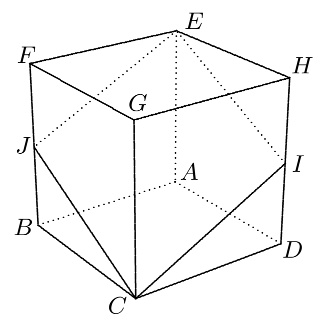
24. (Continuation) Suppose that four such pipes are strapped together with a snugly-fitting band. How long is the band?

25. Which point on the circle $x^2 + y^2 - 12x - 4y = 50$ is closest to the origin? Which point is farthest from the origin? Explain.

26. The lateral edges of a regular hexagonal pyramid are all 20 cm long, and the base edges are all 16 cm long. To the nearest cc, what is the volume of this pyramid? To the nearest square cm, what is the combined area of the base and six lateral faces?

27. There are two circles that go through $(9, 2)$. Each one is tangent to both coordinate axes. Find the center and the radius for each circle. Start by drawing a clear diagram.

28. The figure at right shows a $2 \times 2 \times 2$ cube $ABCDEFGH$, as well as midpoints I and J of its edges DH and BF. It so happens that C, I, E, and J all lie in a plane. Can you justify this statement? What kind of figure is quadrilateral $CIEJ$, and what is its *area*? Is it possible to obtain a polygon with a larger area by slicing the cube with a different plane? If so, show how to do it. If not, explain why it is not possible.



29. Some Exonians bought a circular pizza for \$10.80. Kyle's share was \$2.25. What was the central angle of Kyle's slice?

30. Representing one unit by at least five squares on your graph paper, draw the *unit circle*, which is centered at the origin and goes through point $A = (1, 0)$. Use a protractor to mark the third-quadrant point P on the circle for which arc AP has angular size 215 degrees. Estimate the coordinates of P, reading from your graph paper. Notice that both are negative numbers. Use a calculator to find the cosine and sine values of a 215-degree angle. Explore further to explain why sine and cosine are known as *circular functions*.

31. A plot of land is bounded by a 140-degree circular arc and two 80-foot radii of the same circle. Find the *perimeter* of the plot, as well as its area.

32. Deniz notices that the sun can barely be covered by closing one eye and holding an aspirin tablet, whose diameter is 7 mm, at arm's length, which means 80 cm from Deniz's eye. Find the *apparent size* of the sun, which is the size of the angle *subtended* by the sun.

33. Circles centered at A and B are tangent at T. Prove that A, T, and B are collinear.

34. At constant speed, a wheel makes a full rotation once counterclockwise every 10 seconds. The center of the wheel is fixed at $(0, 0)$ and its radius is 1 foot. A paint spot is initially at $(1, 0)$; where is it t seconds later?

35. The base of a pyramid is the regular polygon $ABCDEFGH$, which has 14-inch sides. All eight of the pyramid's lateral edges, VA, VB, …, VH, are 25 inches long. To the nearest tenth of an inch, calculate the height of pyramid $VABCDEFGH$.

36. (Continuation) To the nearest tenth of a degree, calculate the size of the *dihedral* angle formed by the octagonal base and the triangular face VAB.

37. (Continuation) Points A', B', C', D', E', F', G', and H' are marked on edges VA, VB, VC, VD, VE, VF, VG, and VH, respectively, so that segments VA', VB', …, VH' are all 20 inches long. Find the volume ratio of pyramid $VA'B'C'D'E'F'G'H'$ to pyramid $VABCDEFGH$. Find the volume ratio of *frustum* $A'B'C'D'E'F'G'H'ABCDEFGH$ to pyramid $VABCDEFGH$.

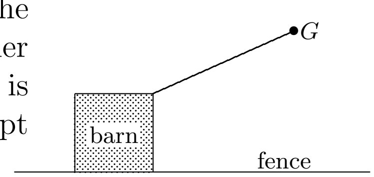
38. Quinn is running around the circular track $x^2+y^2=10000$, whose radius is 100 meters, at 4 meters per second. Quinn starts at the point $(100,0)$ and runs in the counterclockwise direction. After 30 minutes of running, what are Quinn's coordinates?

39. The hypotenuse of a right triangle is 1000, and one of its angles is 87 degrees.
(a) Find the legs and the area of the triangle, correct to three decimal places.
(b) Write a formula for the area of a right triangle in which h is the length of the hypotenuse and A is the size of one of the acute angles.
(c) Apply your formula (b) to redo part (a). Did you get the same answer? Explain.

40. Find the center and the radius for each of the circles $x^2-2x+y^2-4y-4=0$ and $x^2-2x+y^2-4y+5=0$. How many points fit the equation $x^2-2x+y^2-4y+9=0$?

41. What is the result of graphing the equation $(x-h)^2+(y-k)^2=r^2$?

42. Find the total grazing area of the goat G represented in the figure (a top view) shown at right. The animal is tied to a corner of a $40' \times 40'$ barn, by an $80'$ rope. One of the sides of the barn is extended by a fence. Assume that there is grass everywhere except inside the barn.



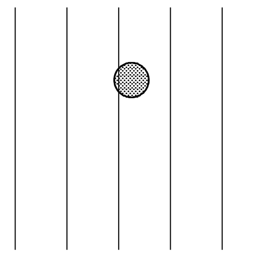
43. A *half-turn* is a 180-degree rotation. Apply the half-turn centered at $(3,2)$ to the point $(7,1)$. Find coordinates of the image point. Find coordinates for the image of (x,y).

44. A 16.0-inch chord is drawn in a circle whose radius is 10.0 inches. What is the angular size of the minor arc of this chord? What is the length of the arc, to the nearest tenth of an inch?

45. What graph is traced by the parametric equation $(x,y)=(\cos t,\sin t)$?

46. What is the area enclosed by a circular *sector* whose radius is r and arc length is s?

47. A coin with a 2-cm diameter is dropped onto a sheet of paper ruled by parallel lines that are 3 cm apart. Which is more likely, that the coin will land on a line, or that it will not?



48. A wheel whose radius is 1 is placed so that its center is at $(3, 2)$. A paint spot on the rim is found at $(4, 2)$. The wheel is spun θ degrees in the counterclockwise direction. Now what are the coordinates of that paint spot?

49. A 36-degree counterclockwise rotation centered at the origin sends the point $A = (6, 3)$ to the image point A'. To three decimal places, find coordinates for A'.

50. In navigational terms, a *minute* is one sixtieth of a *degree*, and a *second* is one sixtieth of a minute. To the nearest foot, what is the length of a one-second arc on the equator? The radius of the earth is 3960 miles.

51. A sector of a circle is enclosed by two 12.0-inch radii and a 9.0-inch arc. Its perimeter is therefore 33.0 inches. What is the area of this sector, to the nearest tenth of a square inch? What is the central angle of the sector, to the nearest tenth of a degree?

52. (Continuation) There is another circular sector — part of a circle of a different size — that has the same 33-inch perimeter and that encloses the same area. Find its central angle, radius, and arc length, rounding to the nearest tenth.

53. Use the unit circle and your knowledge of special triangles to find exact values for $\sin 240$ and $\cos 240$. Then use a calculator to check your answers. Notice that calculators use parentheses around the 240 because sin and cos are functions. In text, except where the parentheses are required for clarity, they are often left out.

54. Given that $\cos 80 = 0.173648\ldots$, explain how to find $\cos 100$, $\cos 260$, $\cos 280$, and $\sin 190$ without using a calculator.

55. Use the unit circle to define $\cos\theta$ and $\sin\theta$ for any number θ between 0 and 360, inclusive. Then explain how to use $\cos\theta$ and $\sin\theta$ to define $\tan\theta$.

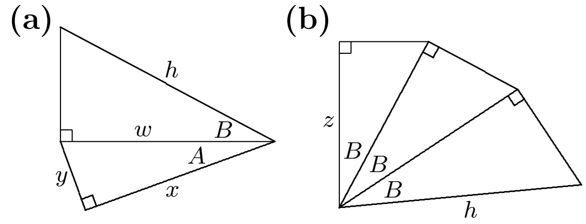
56. Show that your method in the previous question allows you to define $\cos\theta$, $\sin\theta$, and $\tan\theta$ for numbers θ greater than 360 and also for numbers θ less than 0. What do you suppose it means for an angle to be *negative*?

57. A half-turn centered at $(-3, 4)$ is applied to $(-5, 1)$. Find coordinates for the image point. What are the coordinates when the half-turn centered at (a, b) is applied to (x, y)?

58. Translate the circle $x^2 + y^2 = 49$ by the vector $[3, -5]$. Write an equation for the image circle.

59. Point by point, a dilation transforms the circle $x^2 - 6x + y^2 - 8y = -24$ onto the circle $x^2 - 14x + y^2 - 4y = -44$. Find the center and the magnification factor of this transformation.

60. (Continuation) The circles have two common external tangent lines, which meet at the dilation center. Find the size of the angle formed by these lines, and write an equation for each line.



61. Using the figures at right, express the lengths w, x, y, and z in terms of length h and angles A and B.

62. Find at least two values for θ that fit the equation $\sin\theta = \frac{1}{2}\sqrt{3}$. How many such values are there?

63. Choose an angle θ and calculate $(\cos\theta)^2 + (\sin\theta)^2$. Repeat with several other values of θ. Explain the results. It is customary to write $\cos^2\theta + \sin^2\theta$ instead of $(\cos\theta)^2 + (\sin\theta)^2$.

64. What graph is traced by the parametric equation $(x, y) = (2 + \cos t, 1 + \sin t)$?

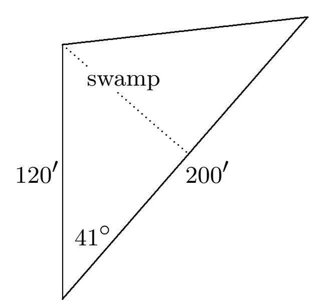
65. A *quarter-turn* is a 90-degree rotation. If the counterclockwise quarter-turn centered at $(3, 2)$ is applied to $(7, 1)$, what are the coordinates of the image? What are the image coordinates when this transformation is applied to a general point (x, y)?

66. A circle centered at the origin meets the line $-7x + 24y = 625$ tangentially. Find coordinates for the point of tangency.

67. Write without parentheses: **(a)** $(xy)^2$ **(b)** $(x + y)^2$ **(c)** $(a\sin B)^2$ **(d)** $(a + \sin B)^2$

68. Transformation $\mathcal{T}$ is defined by $\mathcal{T}(x, y) = (2, 7) + [2 - x, 7 - y]$. An equivalent definition is $\mathcal{T}(x, y) = (4 - x, 14 - y)$. Use the first definition to help you explain what kind of transformation $\mathcal{T}$ is.

69. A 15-degree counterclockwise rotation centered at $(2, 1)$ sends $(4, 6)$ to another point (x, y). Find x and y, correct to three decimal places.



70. A triangular plot of land has the SAS description indicated in the figure shown at right. Although a swamp in the middle of the plot makes it awkward to measure the altitude that is dotted in the diagram, it can at least be calculated. Show how. Then use your answer to find the area of the triangle, to the nearest square foot.

71. (Continuation) Find the length of the third side of the triangle, to the nearest foot.

72. Use the unit circle and, if necessary, a calculator to find all solutions θ for $0 \le \theta \le 360$:
(a) $\cos\theta = -1$ **(b)** $\cos\theta = 0.3420$ **(c)** $\sin\theta = -\frac{1}{2}\sqrt{2}$ **(d)** $\tan\theta = 6.3138$

73. Using the line $y = x$ as a mirror, find the reflected image of the point (a, b). What are the coordinates of the point on the line that is closest to (a, b)?

74. The radius of a circular sector is r. The central angle of the sector is θ. Write formulas for the arc length and the perimeter of the sector.

75. A bird flies linearly, according to the equation $(x, y, z) = (5, 6, 7) + t[2, 3, 1]$. Assume that the sun is directly overhead, making the sun's rays perpendicular to the xy-plane which represents the ground. The bird's shadow is said to be *projected perpendicularly* onto the (level) ground. Find an equation that describes the motion of the shadow.

76. A coin of radius 1 cm is tossed onto a plane surface that has been tesselated (tiled) by rectangles whose measurements are all 8 cm by 15 cm. What is the probability that the coin lands within one of the rectangles?

77. What graph is traced by the parametric equation $(x, y) = (3\cos t, 3\sin t)$? What about the equation $(x, y) = (7 + 3\cos t, -2 + 3\sin t)$?

78. A 15-degree counterclockwise rotation about $(4, 6)$ transforms $(2, 1)$ onto another point (x, y). Find x and y, correct to three decimal places.

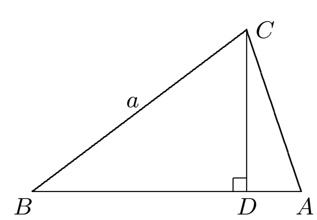
79. Suppose that the lateral faces VAB, VBC, and VCA of triangular pyramid $VABC$ all have the same height drawn from V. Let F be the point in plane ABC that is closest to V, so that VF is the altitude of the pyramid. Show that F is one of the special points of triangle ABC.

80. Simplify: **(a)** $x\cos^2\theta + x\sin^2\theta$ **(b)** $x\cos^2\theta + x\cos^2\theta + 2x\sin^2\theta$

81. A 12.0-cm segment makes a 72.0-degree angle with a 16.0-cm segment. To the nearest tenth of a cm, find the third side of the triangle determined by this SAS information.

82. (Continuation) Find the area of the triangle, to the nearest square centimeter.

83. In the diagram at right, CD is the altitude from C.
(a) Express CD in terms of angle B and side a.
(b) Express BD in terms of angle B and side a.
(c) Simplify the expression $(a\sin B)^2 + (a\cos B)^2$ and discuss its relevance to the diagram.
(d) Why was $a\sin B$ used instead of $\sin B \cdot a$?



84. A 12.0-cm segment makes a 108.0-degree angle with a 16.0-cm segment. To the nearest tenth of a cm, find the third side of the triangle determined by this SAS information.

85. (Continuation) Find the area of the triangle, to the nearest square centimeter.

86. Schuyler has made some glass prisms to be sold as window decorations. Each prism is four inches tall, and has a regular hexagonal base with half-inch sides. They are to be shipped in cylindrical tubes that are 4 inches tall. What radius should Schuyler use for the tubes? Once a prism is inserted into its tube, what volume remains for packing material?

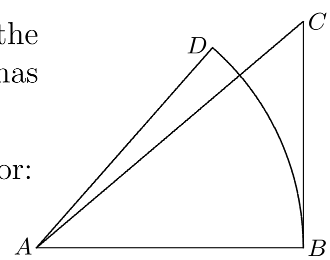
87. Use the unit circle and, if necessary, a calculator to find all solutions t between 360 and 720, inclusive:

(a) $\cos t = \sin t$ **(b)** $\tan t = -4.3315$ **(c)** $\sin t = -0.9397$

88. Find the center and the radius of the circle $x^2 + y^2 - 2ax + 4by = 0$.

89. The wheels of a moving bicycle have 27-inch diameters, and they are spinning at 200 revolutions per minute. How fast is the bicycle traveling, in miles per hour? Through how many degrees does a wheel turn each second?

90. In the figure at right, arc BD is centered at A, and it has the same length as tangent segment BC. Explain why sector ABD has the same area as triangle ABC.



91. Find all solutions A between 0 and 360 without a calculator:

(a) $\cos A = \cos 251$ **(b)** $\cos A = 1.5$

(c) $\sin A = \sin 220$ **(d)** $\cos A = \cos(-110)$

92. Assuming m, n, and p are constants, do all equations of the form $x^2 + mx + y^2 + ny = p$ represent circles? Explain.

93. Find all solutions between 0 and 360 of $\cos t < \frac{1}{2}\sqrt{3}$. Provide a diagram that supports your results.

94. Consider the transformation $\mathcal{T}(x, y) = \left(\frac{4}{5}x - \frac{3}{5}y\,,\ \frac{3}{5}x + \frac{4}{5}y\right)$, which is a rotation centered at the origin. Describe the sequence of points that arise when $\mathcal{T}$ is applied repeatedly, starting with the point $A_0 = (5, 0)$. In other words, A_1 is obtained by applying $\mathcal{T}$ to A_0, then A_2 is obtained by applying $\mathcal{T}$ to A_1, and so forth. Give a detailed description.

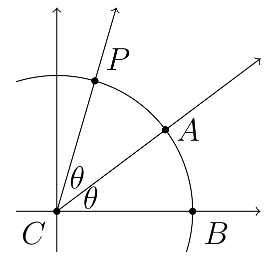
95. How long is the shadow cast on the ground (represented by the xy-plane) by a pole that is eight meters tall, given that the sun's rays are parallel to the vector $[5, 3, -2]$?

96. A conical cup has a 10-cm diameter and is 12 cm deep. How much can this cup hold?

97. (Continuation) Water in the cup is 6 cm deep. What percentage of the cup is filled?

98. (Continuation) Dana takes a paper cone of the given dimensions, cuts it along a straight line from the rim to the vertex, then flattens the paper out on a table. Find the radius, the arc length, and the central angle of the resulting circular sector.

99. Point $A = (\cos\theta, \sin\theta)$ is at the intersection of $x^2 + y^2 = 1$ and a ray starting at the origin that makes an angle, θ, with the positive x-axis. The ray starting at the origin through point P makes an angle of 2θ with the positive x-axis.
(a) Explain why $P = (\cos 2\theta, \sin 2\theta)$.
(b) Reflect $B = (1, 0)$ over the line CA to get an equivalent form of the coordinates of P written in terms of $\cos\theta$ and $\sin\theta$.



100. A javelin lands with six feet of its length sticking out of the ground, making a 52-degree angle with the ground. The sun is directly overhead. The javelin's shadow on the ground is an example of a perpendicular projection. Find its length, to the nearest inch. Henceforth, whenever *projection* appears, "perpendicular" will be understood.

101. The *dot product* of vectors $\mathbf{u} = [a, b]$ and $\mathbf{v} = [m, n]$ is the number $\mathbf{u}\bullet\mathbf{v} = am + bn$. The *dot product* of vectors $\mathbf{u} = [a, b, c]$ and $\mathbf{v} = [m, n, p]$ is the number $\mathbf{u}\bullet\mathbf{v} = am + bn + cp$. In general, the dot product of two vectors is the sum of all the products of corresponding components. Let $\mathbf{u} = [-2, 3, 1]$, $\mathbf{v} = [0, 1, 2]$, and $\mathbf{w} = [1, 2, -1]$. Calculate
(a) $4\mathbf{u}$ **(b)** $\mathbf{u} + \mathbf{v}$ **(c)** $4\mathbf{u} - \mathbf{v}$ **(d)** $\mathbf{u}\bullet(\mathbf{v} + \mathbf{w})$ **(e)** $\mathbf{u}\bullet\mathbf{v} + \mathbf{u}\bullet\mathbf{w}$

102. In triangle ABC, it is given that $BC = 7$, $AB = 3$, and $\cos B = \frac{11}{14}$. Find the length of the projection of **(a)** segment AB onto line BC; **(b)** segment BC onto line AB.

103. Find the coordinates of the shadow cast on the xy-plane by a small object placed at the point $(10, 7, 20)$, assuming that the sun's rays are parallel to the vector $[5, 3, -2]$.

104. Andy is riding a merry-go-round, whose radius is 25 feet and which is turning 36 degrees per second. Seeing a friend in the crowd, Andy steps off the outer edge of the merry-go-round and suddenly finds it necessary to run. At how many miles per hour?

105. The perimeter of a triangle, its area, and the radius of the circle inscribed in the triangle are related in an interesting way. Prove that the radius of the circle times the perimeter of the triangle equals twice the area of the triangle.

106. A circular sector has an 8.26-inch radius and a 12.84-inch arc length. There is another sector that has the same area and the same perimeter. What are its measurements?

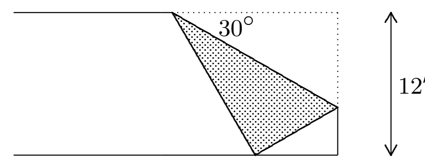
107. (Continuation) Given a circular sector, is there always a different sector that has the same area and the same perimeter? Explain your answer.

108. Solve for y: $x^2 = a^2 + b^2 - 2aby$

109. A segment that is a units long makes a C-degree angle with a segment that is b units long. In terms of a, b, and C, find the third side of the triangle defined by this SAS description. You have done numerical versions of this question. Start by finding the length of the altitude drawn to side b, as well as the length of the perpendicular projection of side a onto side b. The resulting formula is known as the *Law of Cosines*.

110. (Continuation) What is the area of the triangle defined by a, b, and C?

111. The figure at right shows a long rectangular strip of paper, one corner of which has been folded over to meet the opposite edge, thereby creating a 30-degree angle. Given that the width of the strip is 12 inches, find the length of the crease.



112. (Continuation) Suppose that the size of the folding angle is θ degrees. Use trigonometry to express the length of the crease as a function of θ. Check using the case $\theta = 30$.

113. (Continuation) Find approximately that value of θ that makes the crease as short as it can be. Restrict your attention to angles that are smaller than 45 degrees. (Why is this necessary?)

114. A triangle has an 8-inch side, a 10-inch side, and an area of 16 square inches. What can you deduce about the angle formed by these two sides?

115. Jamie rides a Ferris wheel for five minutes. The diameter of the wheel is 10 meters, and its center is 6 meters above the ground. Each revolution of the wheel takes 30 seconds. Being more than 9 meters above the ground enables Jamie to see the ocean. For how many seconds does Jamie see the ocean?

116. (Continuation) What graph is traced by the equation $(x, y) = (5 \sin 12t\,,\ 6 - 5 \cos 12t)$? How do the constants in this equation relate to Jamie's ride? What does t represent?

117. Find two different parametric descriptions for the circle of radius 4 centered at $(-3, 2)$. Use your graphing device to check that your equations produce the same graph.

118. Let $\mathbf{u} = [a, b, c]$, $\mathbf{v} = [p, q, r]$, and $\mathbf{w} = [k, m, n]$ for the following questions:
(a) Verify that $\mathbf{u} \bullet \mathbf{v}$ is the same number as $\mathbf{v} \bullet \mathbf{u}$, for any vectors $\mathbf{u}$ and $\mathbf{v}$.
(b) What is the significance of the number $\mathbf{u} \bullet \mathbf{u}$?
(c) What does the equation $\mathbf{u} \bullet \mathbf{v} = 0$ tell us about the vectors $\mathbf{u}$ and $\mathbf{v}$?
(d) Is it true that $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$ holds for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$?
(e) If $\mathbf{u}$ and $\mathbf{v}$ represent the sides of a parallelogram, $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ represent the diagonals. Justify this. Explain what the equation $(\mathbf{u} + \mathbf{v}) \bullet (\mathbf{u} - \mathbf{v}) = 0$ tells us about the parallelogram. Give an example of nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ that fit this equation.

119. Dana takes a sheet of paper, cuts a 120-degree circular sector from it, then rolls it up and tapes the straight edges together to form a cone. Given that the sector radius is 12 cm, find the height and volume of this paper cone.

120. Find the third side of a triangle in which a 4.00-inch side and a 6.00-inch side are known to make a 56.0-degree angle. Round your answer to three significant digits.

121. Without a calculator, find all solutions w between 0 and 360, inclusive, providing diagrams that support your results.
(a) $\cos w = \cos(-340)$ **(b)** $\cos w = \sin 20$ **(c)** $\sin w = \cos(-10)$
(d) $\sin w < -\frac{1}{2}$ **(e)** $1 < \tan w$

122. The radius of the circumscribed circle of the triangle ABC is 15 cm. Given that B is a 49-degree angle, find the length of side AC.

123. (Continuation) The radius of the circumscribed circle of the triangle ABC is R cm. Given that B is a β-degree angle, find the length of side AC, in terms of R and β.

124. A counterclockwise quarter-turn $\mathcal{Q}$ about the origin is applied to the point (x, y). What are the coordinates of the image point? Answer in the form $\mathcal{Q}(x, y) = (ax + by, cx + dy)$.

125. A coin of radius 1 cm is tossed onto a plane surface that has been tesselated by right triangles whose sides are 8 cm, 15 cm, and 17 cm long. What is the probability that the coin lands within one of the triangles?

126. The table at right shows measurements made on a circle with a one-meter radius. Each entry in the s-column is an arc length, and the adjacent entry in the c-column is the corresponding chord length, both in meters. Explain why $c < s$, and determine the range of values for c and for s. With s on the horizontal axis and c on the vertical axis, sketch an approximate graph of c vs s.

s	c
0.256	0.255
0.618	0.608
1.234	1.157
1.571	1.414
1.896	1.625

127. (Continuation) Express s and c in terms of θ, the central angle that intercepts s and c. Combine these equations to express c as a function of s. Graph this relationship.

128. Given a vector $\mathbf{u}$, the familiar *absolute-value* notation $|\mathbf{u}|$ is often used for its *magnitude*. Thus the expressions $\mathbf{u}\bullet\mathbf{u}$ and $|\mathbf{u}|^2$ both mean the same thing. What exactly *do* they mean? Why do you think absolute value notation is used?

129. For any two numbers a and b, the product of $a - b$ times itself is equal to $a^2 - 2ab + b^2$. Does this familiar algebraic result hold for dot products of a vector $\mathbf{u} - \mathbf{v}$ with itself? In other words, is it true that $(\mathbf{u} - \mathbf{v})\bullet(\mathbf{u} - \mathbf{v}) = \mathbf{u}\bullet\mathbf{u} - 2\mathbf{u}\bullet\mathbf{v} + \mathbf{v}\bullet\mathbf{v}$? Justify your conclusion, trying not to express vectors $\mathbf{u}$ and $\mathbf{v}$ in component form.

130. A triangle has a 56-degree angle, formed by a 10-inch side and an x-inch side. Given that the area of the triangle is 18 square inches, find x.

131. Devon's bike has wheels that are 27 inches in diameter. After the front wheel picks up a tack, Devon rolls another 100 feet and stops. How far above the ground is the tack?

132. An isosceles triangle has two 10-inch sides and a $2w$-inch side. Find the radius of the inscribed circle of this triangle, in the cases $w = 5$, $w = 6$, and $w = 8$.

133. (Continuation) Write an expression for the inscribed radius r in terms of the variable w, then find the value of w, to the nearest hundredth, that gives the maximum value of r.

134. Triangle ABC has a 63.0-degree angle at B, and side AC is 13.6 cm long. What is the diameter of the circle circumscribed about ABC?

135. (Continuation) Given any triangle ABC, with sides a, b, and c opposite angles A, B, and C, respectively, what can be said about the three ratios $\dfrac{a}{\sin A}$, $\dfrac{b}{\sin B}$, and $\dfrac{c}{\sin C}$? This result is known as the *Law of Sines.*

136. Let $\mathbf{u} = [3, -4]$ and $\mathbf{v} = [8, 15]$. Calculate the four numbers $|\mathbf{u}|$, $|\mathbf{v}|$, $|\mathbf{u} - \mathbf{v}|$, and $|\mathbf{u}| - |\mathbf{v}|$. Do the same with the vectors $\mathbf{u} = [2, 6, -3]$ and $\mathbf{v} = [2, 2, 1]$.

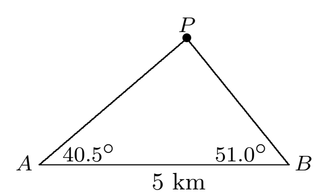
137. An 8-inch tall conical cup has a 6-inch base radius. Explain how to find the surface area of the cup.

138. If a triangle has sides of lengths a and b, which make a C-degree angle, then the length of the side opposite C is c, where $c^2 = a^2 + b^2 - 2ab\cos C$. This is the SAS version of the Law of Cosines. Explain the terminology. Derive an equivalent SSS version of the Law of Cosines, which gives the cosine of the angle in terms of the lengths of the three sides. Now use it to find the angles of the triangle whose sides have lengths 4 cm, 5 cm, and 6 cm.

139. What is the length of the vector $[5\cos\theta, 5\sin\theta]$? If the vector [5,0] is rotated 36 degrees in the counterclockwise direction, what are the components of the resulting vector?

140. Infinitely many different sectors can be cut from a circular piece of paper with a 12-cm radius, and any such sector can be fashioned into a paper cone with a 12-cm *slant height.*
(a) Show that the volume of the cone produced by the 180-degree sector is larger than the volume of the cone produced by the 120-degree sector.
(b) Find a sector of the same circle that will produce a cone whose volume is even larger.
(c) Express the volume of a cone formed from this circle as a function of the central angle of the sector used to form it, then find the sector that produces the cone of greatest volume.

141. Two observers who are 5 km apart simultaneously sight a small airplane flying between them. One observer measures a 51.0-degree inclination angle, while the other observer measures a 40.5-degree inclination angle, as shown in the diagram. At what altitude is the airplane flying?



142. Let $\mathbf{u} = [2, -3, 1]$ and $\mathbf{v} = [0, 1, 4]$. Calculate the vector $\mathbf{u} - \mathbf{v}$. Place $\mathbf{u}$ and $\mathbf{v}$ tail-to-tail to form two sides of a triangle. With regard to this triangle, what does $\mathbf{u} - \mathbf{v}$ represent? Calculate the number $\mathbf{u} \bullet \mathbf{u}$ and discuss its relevance to the diagram you have drawn. Do the same for the number $(\mathbf{u} - \mathbf{v}) \bullet (\mathbf{u} - \mathbf{v})$.

143. A matrix can be used to display and process certain kinds of data. For example, during its final weekend of operation in 2008, the *IOKA* sold tickets to 186 adults on Friday, 109 adults on Saturday, 111 adults on Sunday, 103 children on Friday, 127 children on Saturday, 99 children on Sunday, 77 senior citizens on Friday, 67 senior citizens on Saturday, and 58 senior citizens on Sunday. This data is displayed in the 3×3 sales matrix $\mathbf{S}$ shown above. The descriptive labels given in the margin allow the reader to remember what all the numbers mean. Invent your own example of numerical data that can be displayed like this in a rectangular array.

$$\begin{array}{c} \\ Fr \\ Sa \\ Su \end{array}\begin{array}{c} \begin{array}{ccc} Ch & Ad & Se \end{array} \\ \begin{bmatrix} 103 & 186 & 77 \\ 127 & 109 & 67 \\ 99 & 111 & 58 \end{bmatrix} \end{array} = \mathbf{S}$$

144. (Continuation) The IOKA's ticket prices can be read from the 3×1 matrix $\mathbf{P}$ shown at right. Such a matrix is often called a *column vector*. The first row of matrix $\mathbf{S}$ is a 3-component *row vector*. What is the dot product of these two vectors? What does it mean? What about the dot products of the other rows of $\mathbf{S}$ with $\mathbf{P}$?

$$\begin{array}{c} \\ Ch \\ Ad \\ Se \end{array}\begin{array}{c} \$ \\ \begin{bmatrix} 4.50 \\ 7.50 \\ 6.00 \end{bmatrix} \end{array} = \mathbf{P}$$

145. (Continuation) *Matrix multiplication* consists of calculating all possible dot products of row vectors from the first matrix and column vectors from the second matrix. How many dot products can be formed by multiplying matrix $\mathbf{S}$ times matrix $\mathbf{P}$? How would you organize them into a new matrix, called $\mathbf{SP}$? What do the entries of $\mathbf{SP}$ mean?

146. The lengths of segments PQ and PR are 8 inches and 5 inches, respectively, and they make a 60-degree angle at P.
(a) Find the area of triangle PQR.
(b) Find the length of the projection of segment PQ onto segment PR.
(c) Find the length of segment QR.
(d) Find the sizes of the other two angles of triangle PQR.
(e) Find the length of the median drawn to side PQ.
(f) Find the length of the bisector of angle R.
(g) Find the third side of another triangle that has a 5-inch side, an 8-inch side, and the same area as triangle PQR.

147. Draw the unit circle and a first-quadrant ray from the origin that makes an angle θ with the positive x-axis. Let B be the point on this ray whose x-coordinate is 1, and let $A = (1, 0)$. Segment AB is tangent to the circle. In terms of θ, find its length. Hmm ...

148. Triangle PEA has a 20-degree angle at P and a 120-degree angle at E, and the length of side EA is 6 inches. Find the lengths of the other two sides of this triangle.

149. If two vectors $\mathbf{u}$ and $\mathbf{v}$ fit the equation $(\mathbf{u} - \mathbf{v}) \bullet (\mathbf{u} - \mathbf{v}) = \mathbf{u} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v}$, how must these vectors $\mathbf{u}$ and $\mathbf{v}$ be related? What familiar theorem does this equation represent?

150. The lengths QR, RP, and PQ in triangle PQR are often denoted p, q, and r, respectively. What do the formulas $\frac{1}{2}pq \sin R$ and $\frac{1}{2}qr \sin P$ mean? After you justify the equation $\frac{1}{2}pq \sin R = \frac{1}{2}qr \sin P$, simplify it to a familiar form.

151. The price of a large pepperoni pizza is \$13.45 at New England Pizza, \$13.50 at Romeo's, and \$15 at Front Row. These shops charge \$6.25, \$10.50, and \$8, respectively, for a Greek salad. What would the bill be at each shop for seven pizzas and five salads?

152. (Continuation) What would each shop charge for two pizzas and a dozen salads? Show how this problem, as well as the previous one, can be solved by forming two suitable matrices and then multiplying them.

153. If two angles are supplementary, then their sines are equal. Explain why. What about the cosines of supplementary angles? If you are not sure, calculate some examples.

154. An isosceles triangle has two sides of length w that make a 2α-degree angle. Write down two different formulas for the area of this triangle, in terms of w and α (*Greek* "alpha"). By equating the formulas, discover a relation involving $\sin 2\alpha$, $\sin\alpha$, and $\cos\alpha$.

155. A parallelogram has a 7-inch side and a 9-inch side, and the longer diagonal is 14 inches long. Find the length of the other diagonal. Do you need a calculator to do it?

156. (Continuation) Write an equation that expresses the relationship between the side lengths, a and b, and the diagonals, d_1 and d_2. One method is to use the Law of Cosines.

157. Multiplying two matrices consists of calculating several dot products and then arranging them to form a new matrix. There is a natural way to arrange these dot products, each of which combines a row vector from the first matrix and a column vector from the second matrix. Use the example as a model to calculate the matrix products that follow.

$$\text{Example:} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 0 \end{bmatrix} = \begin{bmatrix} 21 & 24 & 7 \\ 47 & 54 & 21 \end{bmatrix}$$

(a) $\begin{bmatrix} 2 & -3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 4 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ -3 & 4 \end{bmatrix}$ **(c)** $\begin{bmatrix} 2 & -3 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ -7 \end{bmatrix}$

158. Describe all the points on the earth's surface that are exactly 4000 miles from the North Pole. Remember that the radius of the earth is 3960 miles.

159. In each of the following, find the angle formed by $\mathbf{u}$ and $\mathbf{v}$:
(a) $\mathbf{u} = [2, 1]$ and $\mathbf{v} = [1, -3]$ **(b)** $\mathbf{u} = [-1, 0, 1]$ and $\mathbf{v} = [0, 2, -2]$

160. Given a triangle ABC in which angle B is exactly twice the size of angle C, *must* it be true that side AC is exactly twice the size of side AB? *Could* it be true?

161. There are two noncongruent triangles that have a 9-inch side, a 10-inch side, and that enclose 36 square inches of area. Find the length of the third side in each of these triangles.

162. Centered 6 meters above the ground, a Ferris wheel of radius 5 meters rotates at 1 degree per second. Assuming that Jamie's ride begins at the lowest point on the wheel, find how far Jamie is above the ground after 29 seconds; after 331 seconds; after t seconds.

163. (Continuation) Graph the equation $y = 6 - 5\cos x$. What does this picture tell you about Jamie's ride? Would a graph of $y = 6 + 5\cos x$ mean anything?

164. Draw vectors $\mathbf{u}$ and $\mathbf{v}$ tail-to-tail so that they make a θ-degree angle. Draw the vector $\mathbf{u} - \mathbf{v}$, the third side of the triangle, and check to see that it points in the right direction.
(a) Solve for $\cos\theta$ using the SSS version of the Law of Cosines, expressing all lengths in terms of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$.
(b) If you use vector algebra to simplify the numerator as much as possible, you will discover an interesting new result connecting $\mathbf{u}\bullet\mathbf{v}$ to $\cos\theta$.

165. A paper cone has an e-inch slant height and an r-inch base radius. In terms of the quantities e and r, write a formula for the *lateral area* of the cone. In other words, find the area of the circular sector obtained by cutting the cone from base to vertex and flattening it out.

166. Let $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{N} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{P} = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Evaluate all possible two-matrix products.

167. Find the angle formed when $[3, 4]$ and $[-5, 12]$ are placed tail-to-tail, then find components for the *vector projection* that results when $[3, 4]$ is projected onto $[-5, 12]$.

168. Two matrices can be multiplied only if their sizes are compatible. Suppose that $\mathbf{U}$ is an $m \times n$ matrix, and that $\mathbf{V}$ is a $p \times q$ matrix. In order for the matrix product $\mathbf{UV}$ to make sense, what must be true about the dimensions of these matrices?

169. A sphere consists of all the points that are 5 units from its center $(2, 3, -6)$. Write an equation that describes this sphere. Does the sphere intersect the xy-plane? Explain.

170. The curve $(x-1)^2 + (y-2)^2 = 9$ can be parameterized in many different ways. Investigate the differences in the following.
(a) $(x, y) = (1 + 3\cos t, 2 + 3\sin t)$
(b) $(x, y) = (1 + 3\sin t, 2 + 3\cos t)$
(c) $(x, y) = (1 - 3\sin t, 2 + 3\cos t)$
(d) $(x, y) = (1 + 3\cos 2t, 2 + 3\sin 2t)$

171. Sam owns a triangular piece of land on which the tax collector wishes to determine the correct property tax. Sam tells the collector that "the first side lies on a straight section of road and the second side is a stone wall. The wall meets the road at a 24-degree angle. The third side of the property is formed by a 180-foot-long fence, which meets the wall at a point that is 340 feet from the corner where the wall meets the road." After a little thought, the tax collector realizes that Sam's description of his property is ambiguous, because there are still two possible lengths for the first side. By means of a clear diagram, explain this situation, and calculate the two possible areas, to the nearest square foot.

172. What does the graph of $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ look like?

173. Assuming $q = 6$ and angle $P = 30$ degrees, for each of the following, tell how many noncongruent triangles PQR fit the given description, and find the size of angle Q.
(a) $p = 4$ **(b)** $p = 8$ **(c)** $p = 2$ **(d)** $p = 3$

174. The wheels on Devon's bike have r-inch radii. After the front wheel picks up a tack, Devon rolls another d feet and stops. How far above the ground is the tack?

175. Describe the effect of each of the following geometric transformations. To generate and test your hypotheses, transform some simple points.
(a) $\mathcal{T}(x, y) = (-3x, -3y)$ **(b)** $\mathcal{T}(x, y) = (-y, x)$
(c) $\mathcal{T}(x, y) = (-y, -x)$ **(d)** $\mathcal{T}(x, y) = (0.6x - 0.8y, 0.8x + 0.6y)$

176. (Continuation) Each transformation takes the general form $\mathcal{T}(x, y) = (ax+by, cx+dy)$, using suitable constants a, b, c, and d. For example, $a = -3$, $b = 0 = c$, and $d = -3$ in part (a). What are the values of a, b, c, and d for the remaining examples?

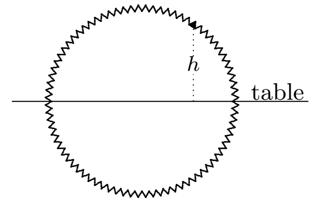
177. (Continuation) Since the expression $ax+by$ is equivalent to the dot product $[a, b] \bullet [x, y]$, matrices can be used to represent these transformations. Explain the connection between the matrix product $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ and the transformation $\mathcal{T}(x, y) = (ax + by, cx + dy)$. Write the *coefficient matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for each of the transformations above.

178. Find the angle formed when $[4, 4, 2]$ and $[4, 3, 12]$ are placed tail-to-tail; then find the components of the vector that results when $[4, 3, 12]$ is projected onto $[4, 4, 2]$.

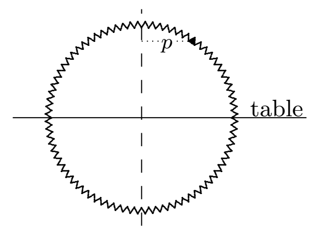
179. Consider again the sphere of radius 5 centered at $(2, 3, -6)$. Describe the intersection of the sphere with the xz-plane. Write an equation (or equations) for this curve.

180. *The triangle inequality.* Explain why $|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$ holds for any vectors $\mathbf{u}$ and $\mathbf{v}$.

181. A large circular saw blade with a 1-foot radius is mounted so that exactly half of it shows above the table. It is spinning slowly, at one degree per second counterclockwise. One tooth of the blade has been painted red. This tooth is initially 0 feet above the table, and rising. What is the height after 37 seconds? After 237 seconds? After t seconds? Draw by hand a graph that shows how the height h of the red tooth is determined by the elapsed time t. It is customary to say that h is a function of t.



182. (Continuation) Now explore the position of the red saw tooth in reference to an imaginary vertical axis of symmetry of the circular blade. The red tooth is initially one foot to the right of the dotted line. How far to the right of this axis is the tooth after 37 seconds? After 237 seconds? After t seconds? Draw by hand a graph that shows how the displacement p of the red tooth with respect to the vertical axis is a function of the elapsed time t.



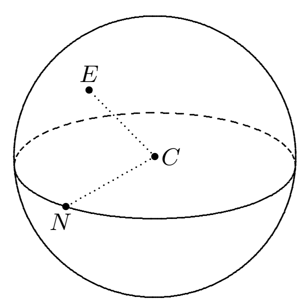
183. (Continuation) The graphs of the height h and the horizontal displacement p of the red saw tooth are examples of *sine* and *cosine* curves, respectively. Draw $y = \sin x$ and $y = \cos x$ on a graphing tool and compare them with the graphs that you drew in the preceding exercises. Use these graphs to answer the following questions:
(a) For what values of t is the red tooth 0.8 feet above the table? 0.8 feet below the table?
(b) When is the tooth 6 inches to the right of the vertical axis? When is it farthest left?

184. Asked to simplify the expression $\sin(180 - \theta)$, Rory volunteered the following solution: $\sin(180 - \theta) = \sin 180 - \sin\theta$, and, because $\sin 180$ is zero, it follows that $\sin(180 - \theta)$ is the same as $-\sin\theta$. Is this answer correct? If not, what is a correct way to express $\sin(180 - \theta)$ in simpler form? Answer the same question for $\cos(180 - \theta)$.

185. Find simpler, equivalent expressions for the following. Justify your answers.
(a) $\sin(180 + \theta)$ **(b)** $\cos(180 + \theta)$ **(c)** $\tan(180 + \theta)$

186. Show that there are at least two ways to calculate the angle formed by the vectors $[\cos 19, \sin 19]$ and $[\cos 54, \sin 54]$.

187. Let N be the point on the equator closest to E = Exeter, and let C be the center of the earth. The central angle ECN is called the *latitude* of E; it is approximately 43 degrees. Take the radius of the earth to be 3960 miles as you answer the following distance questions:
(a) How far from the equator is Exeter? Travel *on* the earth, not through it.
(b) How far does the earth's rotation on its axis carry the citizens of Exeter during a single day?



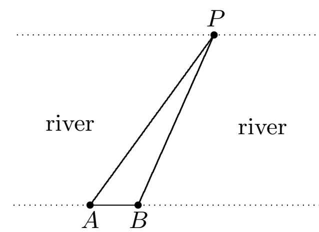
188. The matrix $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ represents the transformation $\mathcal{T}(x, y) = (3x - 4y, 4x + 3y)$.

(a) Apply transformation $\mathcal{T}$ to the *unit square* whose vertices are $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$. In particular, notice what the images of the points $(1, 0)$ and $(0, 1)$ are, and compare them with the entries in the columns of the coefficient matrix.

(b) Confirm that the same results can be obtained by doing some matrix arithmetic: Calculate products $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and interpret.

189. Calculate the products $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and interpret the results.

190. Two industrious PEA students are trying to find the distance across the Squamscott River. They mark points A and B on one bank, 60 meters apart. Then they sight the Powderhouse P on the opposite bank, and measure angles PAB and PBA to be 54 and 114 degrees, respectively. This enables them to calculate the altitude from P to the baseline AB. To the nearest meter, what is their result?



191. If $\sin A$ is known to be 0.96, then what can be said about $\cos A$? What if it is also known that angle A is obtuse?

192. Find components for the vector projection of $[12, 5]$ onto $[-9, 12]$.

193. During one term in Math 310, Min Lee took seven tests, the last of which carried twice the weight of each of the others when averages were computed. Min's test-score vector for the term was $[84, 78, 91, 80, 72, 88, 83]$. Show that Min's final average, a *weighted average*, can be calculated as a dot product of this vector with another seven-component vector. How can the teacher obtain a class list of test averages by multiplying two matrices?

194. Two fire wardens are stationed at locations P and Q, which are 45.0 km apart. Each warden sights the forest fire at F. Given that angle FPQ is 52.0 degrees and angle FQP is 43.0 degrees, find the distance from F to the nearer warden, to the nearest 0.1 km.

195. Find the area of a triangle that has a 10-inch side, a 17-inch side, and a 21-inch side.

196. For each of the following coefficient matrices, describe the effect of its transformation:

(a) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 5/13 & -12/13 \\ 12/13 & 5/13 \end{bmatrix}$ **(d)** $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

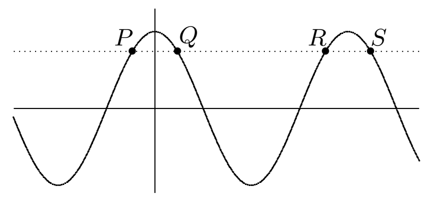
197. (Continuation) Which of the preceding matrices represent isometries? In order for a matrix to represent an isometry, what must be true of its column vectors?

198. Use the vector form of the Law of Cosines to show that $|\mathbf{u} + \mathbf{v}|^2 \le (|\mathbf{u}| + |\mathbf{v}|)^2$ holds for any vectors $\mathbf{u}$ and $\mathbf{v}$. What does this prove?

199. Apply the 57-degree counterclockwise rotation about the origin to the vectors $[1,0]$ and $[0,1]$, then use the image vectors (*written as columns*) to form the coefficient matrix $\mathbf{M}$ for the rotation. Test $\mathbf{M}$ by calculating the products $\mathbf{M}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{M}\begin{bmatrix}0\\1\end{bmatrix}$. Where does this rotation send the vector $[3,1]$? Does $\mathbf{M}$, when applied to $\begin{bmatrix}3\\1\end{bmatrix}$, do its job correctly?

200. Write the coefficient matrix for a θ-degree counterclockwise rotation about the origin.

201. Let $A=(0,0,0)$, $B=(9,8,12)$, and $C=(6,2,3)$. Find coordinates for the point on line AB that is closest to C.

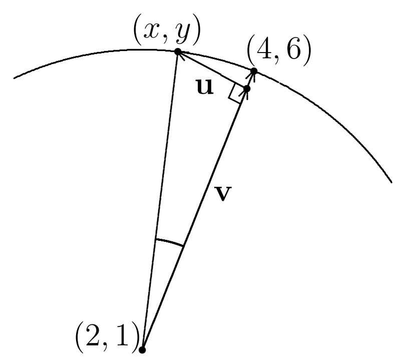


202. At right you see the graphs of $y=\cos x$ and $y=0.7431$ (dotted). Given that $Q=(42,0.7431)$, find coordinates for the intersection points P, R, and S without using a calculator. Use a calculator to check your answers.

203. Verify that the circles $x^2+y^2=25$ and $(x-5)^2+(y-10)^2=50$ intersect at $A=(4,3)$. Find the size of the acute angle formed at A by the intersecting circles. You will first have to decide what is meant by the phrase *the angle formed by the intersecting circles*. To do so, graph the circles on a graphing device and zoom in on point A. What do you notice?

204. (Continuation) The circles intersect at a second point B. Find coordinates for B. What can be said about the angle of intersection formed by the circles at B?

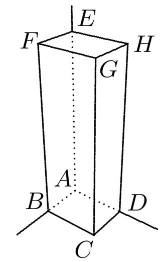
205. Let $A=(-7,-4)$ and $B=(7,4)$, and consider the equation $\overrightarrow{PA}\bullet\overrightarrow{PB}=0$. Describe the configuration of all points $P=(x,y)$ that solve this equation.



206. Previously you found that a 15-degree counterclockwise rotation centered at $(2,1)$ sends the point $(4,6)$ to another point $(x,y)\approx(2.638,6.347)$. The diagram on the right shows the vector $\mathbf{v}$ in the same direction as vector $[2,5]$, only with a different length. Similarly, the vector $\mathbf{u}$ is perpendicular to $\mathbf{v}$ and a different length. Show how you can use $(2,1)+\mathbf{v}+\mathbf{u}$ to find the rotated point (x,y).

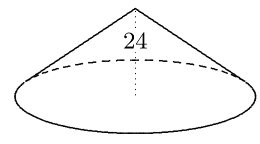
207. (Continuation) Show how a coefficient matrix with a 15-degree counterclockwise rotation can be used to find the coordinates for the point (x,y). Explain how this is (nearly) equivalent to the vector expression $(2,1)+\mathbf{v}+\mathbf{u}$.

208. The diagram at right shows a rectangular solid, two of whose vertices are $A = (0, 0, 0)$ and $G = (3, 4, 12)$. Find angle FBH and the vector projection of $\overrightarrow{BF}$ onto $\overrightarrow{BH}$.



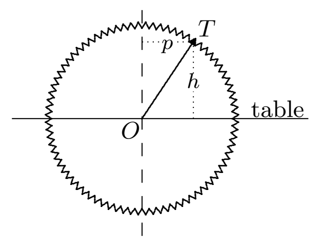
209. To the nearest tenth of a degree, find at least three solutions to each of the following:
(a) $\sin A = 0.80902$ **(b)** $\cos B = -0.80902$ **(c)** $\tan C = 1.96261$.

210. The highway department keeps its sand in a conical storage building that is 24 feet high and 64 feet in diameter. To estimate the cost of painting the building, the lateral area of the cone is needed. To the nearest square foot, what is the area?



211. Let vectors $\mathbf{u}$ and $\mathbf{v}$ form an angle θ when placed tail-to-tail, and let $\mathbf{w}$ be the vector projection of $\mathbf{v}$ onto $\mathbf{u}$.
(a) Assume that θ is acute. Notice that $\mathbf{w}$ points in the same direction as $\mathbf{u}$. Find $|\mathbf{w}|$, then show that $\mathbf{w} = |\mathbf{v}| \dfrac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \dfrac{1}{|\mathbf{u}|} \mathbf{u}$, which simplifies to just $\mathbf{w} = \dfrac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{u}\bullet\mathbf{u}} \mathbf{u}$.
(b) If θ is obtuse, do $\mathbf{w}$ and $\mathbf{u}$ point in the same direction? Does formula (a) still work? By the way, the notation $\mathbf{proj_u v}$ is sometimes used for the vector projection of $\mathbf{v}$ onto $\mathbf{u}$.

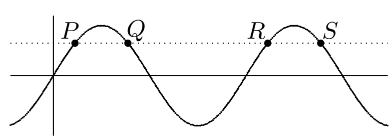
212. Revisit the 1-foot radius circular saw blade with the one red tooth. Look at the ratio m of the height h to the horizontal displacement p. (So $m = h/p$.) The red tooth starts at the rightmost point of the saw and rotates at one degree per second counterclockwise. What is m after 37 seconds? After 137 seconds? After 237 seconds? After t seconds? Draw by hand a graph that shows how m is a function of the elapsed time t. What does the ratio $m = h/p$ tell you about the line OT from the saw center to the tooth?



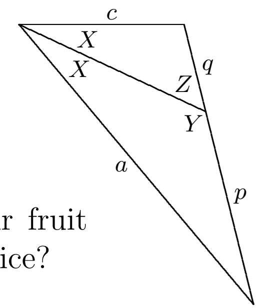
213. (Continuation) The graph of m versus t is an example of a *tangent* curve. Draw $y = \tan x$ on a graphing tool and compare it to the graph you drew in the previous exercise. Use this graph to determine the values of t for which m takes on the following values: 0, 0.5, and -2. How large can m be? Is m defined for *all* values of t?

214. Find the entries of the following matrices:
(a) the 2×2 matrix $\mathbf{M}$ for the reflection across the line $y = x$.
(b) the 2×2 matrix $\mathbf{N}$ for the 90-degree counterclockwise rotation about the origin.
(c) the product $\mathbf{MN}$; what transformation does this represent?
(d) the product $\mathbf{NM}$; what transformation does this represent?
(e) the product $\mathbf{MM}$; what transformation does this represent?

215. The graphs of $y = \sin x$ and $y = k$ (dotted) are shown at right. Given that the coordinates of P are (θ, k), find the coordinates of Q, R, and S, in terms of θ and k.



216. Refer to the diagram at right for the following questions: Express the ratio $p : a$ in terms of $\sin X$ and $\sin Y$. Express the ratio $q : c$ in terms of $\sin X$ and $\sin Z$. Because angles Y and Z are supplementary, you can now combine the preceding answers to obtain a familiar result about angle bisectors.



217. The diameter of a spherical grapefruit is 6.0 inches. An amateur fruit slicer misses its center by one inch. What is the radius of the circular slice?

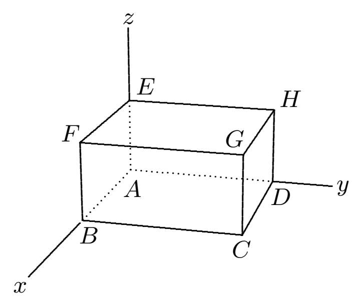
218. The point $P = (-5, 8)$ is in the second quadrant. You are used to describing it by using the *rectangular coordinates* -5 and 8. It is also possible to accurately describe the location of P by using a different pair of coordinates, $P = (r; \theta)$, called *polar coordinates.* The *distance from the origin* is r and θ is an angle in *standard position.* Calculate polar coordinates for P, and notice that there is more than one correct answer.

219. Verify that $\begin{bmatrix} \cos 32 & -\sin 32 \\ \sin 32 & \cos 32 \end{bmatrix} \begin{bmatrix} \cos 40 & -\sin 40 \\ \sin 40 & \cos 40 \end{bmatrix} = \begin{bmatrix} \cos 72 & -\sin 72 \\ \sin 72 & \cos 72 \end{bmatrix}$, and explain why this result could have been expected.

220. The diagram at right shows a rectangular solid, two of whose vertices are $A = (0, 0, 0)$ and $G = (4, 6, 3)$.

(a) Find vector projections of $\overrightarrow{AG}$ onto $\overrightarrow{AB}$, $\overrightarrow{AD}$, and $\overrightarrow{AE}$.

(b) Find the point on segment AC that is closest to the midpoint of segment GH.



221. The result of reflecting across the line $y = -x$ and then rotating 330 degrees counterclockwise around the origin is an isometry $\mathcal{T}$. Represent $\mathcal{T}$ by a 2×2 matrix. There is more than one way to do it. Use the point $(1, 1)$ to check your answer.

222. Given points A and B in 3-dimensional space, describe the solutions to $\overrightarrow{PA} \bullet \overrightarrow{PB} = 0$.

223. To win the carnival game *Ring Ding*, you must toss a wooden ring onto a grid of rectangles so that it lands without touching any of the grid lines. The ring has a 3-inch diameter, the rectangles are twice as long as they are wide, and the game has been designed so that you have a 28% chance of winning. What are the dimensions of each rectangle?

224. Simplify the following:

(a) $\cos(360 - \theta)$ **(b)** $\sin(360 - \theta)$ **(c)** $\cos(360 + \theta)$
(d) $\sin(360 + \theta)$ **(e)** $\tan(360 + \theta)$

225. A sphere of radius r inches is sliced by a plane that is k inches from the center. In terms of r and k, what is the radius of the circle of intersection?

226. A 36-degree counterclockwise rotation centered at the origin sends the point $A = (6, 3)$ to the image point A'. Use a rotation matrix to find coordinates for A'.

227. Calculate the product $\begin{bmatrix} \cos 47 & -\sin 47 \\ \sin 47 & \cos 47 \end{bmatrix} \begin{bmatrix} \cos 47 & \sin 47 \\ -\sin 47 & \cos 47 \end{bmatrix}$. Interpret the result.

228. A parallelogram has a 5-inch side and an 8-inch side that make a 50-degree angle. Find the area of the parallelogram and the lengths of its diagonals.

229. Let $\mathbf{M}$ be the matrix $\begin{bmatrix} 13 & 7 \\ 5 & 3 \end{bmatrix}$. Use a calculator to evaluate $\mathbf{MM}^{-1}$. The matrix $\mathbf{M}^{-1}$ is called the *inverse* of matrix $\mathbf{M}$. Choose another square matrix $\mathbf{M}$ and repeat.

230. Find the diameter of the circle that can be circumscribed around a triangle that has two 13-inch sides and one 10-inch side.

231. What does the SAS version of the Law of Cosines have to say about the "triangle" whose sides p and q form a 180-degree angle?

232. For each of the following, calculate $\mathbf{MN}$ and $\mathbf{NM}$:

(a) $\mathbf{M} = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$, $\mathbf{N} = \begin{bmatrix} \frac{1}{11} & \frac{2}{11} \\ -\frac{3}{11} & \frac{5}{11} \end{bmatrix}$ **(b)** $\mathbf{M} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$, $\mathbf{N} = \begin{bmatrix} 0 & -1 & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -1 & -2 \end{bmatrix}$

(c) $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{N} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$

233. (Continuation) The matrix that results each time that $\mathbf{M}$ is multiplied by $\mathbf{N}$ is called the *identity matrix* $\mathbf{I}$, and $\mathbf{N}$ is usually written as $\mathbf{M}^{-1}$. Without using a calculator, find $\mathbf{M}^{-1}$ for each of the matrices $\mathbf{M} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{M} = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$.

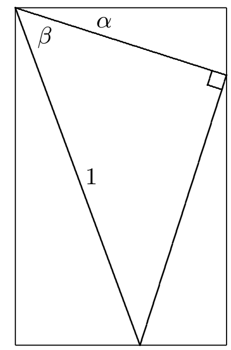
234. Polar coordinates for a point P in the xy-plane consist of two numbers, r and θ, where r is the distance from P to the origin O, and θ is the size of an angle in standard position that has OP as its terminal ray. Find polar coordinates for each of the following points:
(a) $(0, 1)$ **(b)** $(-1, 1)$ **(c)** $(4, -3)$ **(d)** $(1, 7)$ **(e)** $(-1, -7)$

235. Triangle KLM has a 120-degree angle at K and side KL is three fourths as long as side LM. To the nearest tenth of a degree, find the sizes of the other two angles of KLM.

236. A drinking cup is 27/64 full of liquid. What is the ratio of the depth of the liquid to the depth of the cup, assuming that **(a)** the cup is cylindrical? **(b)** the cup is conical?

237. A kite has a 6.00-inch side and a 13.00-inch side, and one of the diagonals is 15.00 inches long. Find the length of the other diagonal, to the nearest hundredth of an inch.

238. A jet leaves Oslo, whose latitude is 60 degrees north of the equator, and flies due west until it returns to Oslo. How far does the jet travel? The radius of the earth is 3960 miles.



239. The rectangle shown has been formed by fitting together four right triangles. As marked, the sizes of two of the angles are α and β (*Greek* "alpha" and "beta"), and the length of one segment is 1. Find the two unmarked angles whose sizes are α and $\alpha+\beta$. By labeling all the segments of the diagram, discover formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, written in terms of $\sin\alpha$, $\cos\alpha$, $\sin\beta$, and $\cos\beta$.

240. A ladder leans against a side of a building, making a 63-degree angle with the ground, and reaching over a fence that is 6 feet from the building. The ladder barely touches the top of the fence, which is 8 feet tall. Find the length of the ladder.

241. The line defined by $(x, y, z) = (2 + t, 10 - 2t, 6 - t)$ intersects the line defined by $(x, y, z) = (9 + 2u, 2 - u, 5 + u)$. Find coordinates for the point of intersection. To the nearest degree, find the size of the acute angle formed by these lines.

242. Consider the points $A = (\cos 40, \sin 40)$, $B = (\cos 76, \sin 76)$, $C = (\cos 121, \sin 121)$, and $D = (\cos 157, \sin 157)$. Find the lengths of segments AB and CD, then explain what is predictable about the answer.

243. Find the volume of a cone of height 8 centimeters and base radius 6 centimeters. This cone is sliced by a plane that is parallel to the base and 2 centimeters from it. Find the volumes of the two resulting solids. One is a cone, while the other is called a *frustum*.

244. For what values of θ is it true that $\sin\theta = \cos\theta$?

245. Points $A = (-5, 12, 0)$ and $B = (13, 0, 0)$ are on the sphere $x^2 + y^2 + z^2 = 13^2$. Find the distance that separates A and B, traveling *on* the sphere, not through it.

246. Points $C = (3, 4, 12)$ and $D = (12, 3, 4)$ are on the sphere $x^2 + y^2 + z^2 = 13^2$. Find the distance that separates C and D, traveling *on* the sphere, not through it.

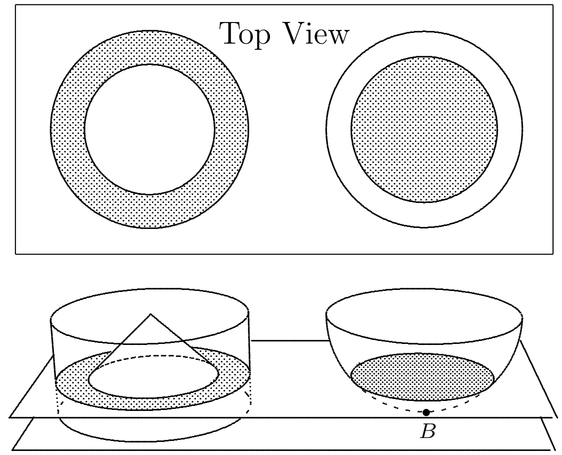
247. A triangle has a 13-inch side, a 14-inch side, and a 15-inch side. To the nearest tenth of an inch, how long is the median drawn to the 14-inch side?

248. In January and February, Herbie's Calculator Shop recorded the sales data shown below in the right-hand 3×2 matrix **S**. The prices for these models are shown below in the left-hand 1×3 matrix **P**. What is the meaning of the entries in the matrix product **PS**? Does the matrix product **SP** make sense?

$$\begin{array}{r} \\ \textit{Price} \end{array}\begin{array}{c} \begin{array}{ccc} \textit{TI-84} & \textit{TI-89} & \textit{TI-Nspire} \end{array} \\ \begin{bmatrix} 85 & 105 & 160 \end{bmatrix} \end{array} \qquad \begin{array}{r} \\ \textit{TI-84} \\ \textit{TI-89} \\ \textit{TI-Nspire} \end{array}\begin{array}{c} \begin{array}{cc} \textit{Jan} & \textit{Feb} \end{array} \\ \begin{bmatrix} 12 & 7 \\ 34 & 22 \\ 56 & 39 \end{bmatrix} \end{array}$$

249. (Continuation) Herbie decides to lower all calculator prices by 10%. Show how this can be done by multiplying one of the matrices above by a suitable scalar.

250. A cylinder rests on top of a table, with a cone inscribed within, vertex up. Both heights and radii are 8 cm. A hemispherical bowl of radius 8 cm rests nearby on the same table, its circular rim parallel to the table.



Consider that part of the cylinder that is outside (above) the cone. Slice this region by a plane that is parallel to the table and 3 cm from it. The intersection is a *ring* between two concentric circles. Calculate its area.

The same plane slices the hemisphere, creating a disk. Show that the disk has the same area as the ring. The diagram shows both a top view and a side view of the ring, the disk, and the hemisphere. Note that point B is on the table.

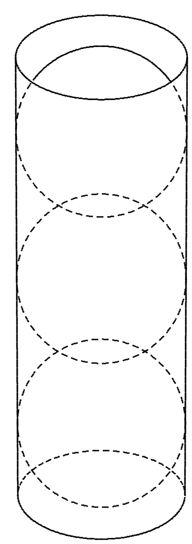
251. (Continuation) Let r be the common radius of the cone, cylinder, and hemisphere. Show that *the ring and the disk have the same area, for all positions of the slicing plane.*

252. (Continuation) If the hemispherical bowl were filled with liquid, it could be poured into the cylinder, which still has the cone inscribed in it. Will all the liquid fit? Expressed in terms of r, what is the volume of the cone? of the empty cylinder? of the hemisphere?

253. (Continuation) Show that a sphere of radius r encloses a volume of $\frac{4}{3}\pi r^3$.

254. Three softball teams ordered equipment from the same catalog. The first team spent \$285 on 5 shirts, 4 caps, and 8 bats. The second team spent \$210 on 12 shirts and 6 caps. The third team spent \$250 for 7 shirts, 10 caps, and 3 bats. What were the catalog prices for shirts, caps, and bats?

255. Find two equivalent ways to express the slope of the vector $[\cos\theta, \sin\theta]$.



256. Describe the points on the earth's surface that are visible to a viewer who is 100 miles above the North Pole.

257. Points P and Q on the unit circle are reflected images of each other, using the y-axis as a mirror. Suppose that P is described by the angle θ; what angle describes Q? In terms of θ, what are the rectangular coordinates of P? Simplify the expressions $\cos(180 - \theta)$ and $\sin(180 - \theta)$ by finding two different ways of writing the rectangular coordinates for Q.

258. A conical cup is 64/125 full of liquid. What is the ratio of the depth of the liquid to the depth of the cup? Conical cups appear fuller than cylindrical cups — explain why.

259. Three tennis balls fit snugly inside a cylindrical can. What percent of the available space inside the can is occupied by the balls?

260. Seward, Alaska is 180 degrees due west of St. Petersburg, Russia. Both cities are 60 degrees north of the equator. Calculate the distance from St. Petersburg to Seward, assuming
(a) we travel along the circle of latitude;
(b) we travel along the circle that passes over the North Pole.
Part (b) is an example of a *great-circle* route. Explain the terminology, and also explain why pilots might prefer to fly along great circles.

261. What does the SAS version of the Law of Cosines have to say about the "triangle" whose sides p and q form a 0-degree angle?

262. In an effort to make their product seem like a better bargain, the Chock-a-Lot candy company increased the size of their chocolate balls, from a 2-cm diameter to a 3-cm diameter, without increasing the price. In fact, the new balls still contain the same amount of chocolate, because they are *hollow spherical shells*, while the 2-cm balls are solid chocolate. How thick are the spherical chocolate shells that Chock-a-Lot is now selling?

263. You have perhaps noticed that any linear equation can be interpreted as a dot-product equation. For example, $3x - 5y = 17$ is equivalent to $[3, -5] \bullet [x, y] = 17$. In this way, any system of linear equations can be written as a single matrix equation. For example,

$$\begin{cases} 2x - 5y + 4z = -7 \\ 6x - y + 3z = 12 \\ -5x + 4y + z = 3 \end{cases} \quad \text{can be written as} \quad \begin{bmatrix} 2 & -5 & 4 \\ 6 & -1 & 3 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 12 \\ 3 \end{bmatrix}$$

Rewrite each of the following systems in matrix form:

$$\textbf{(a)} \begin{cases} 5x + 3y = 15 \\ 4x - 2y = 1 \end{cases} \qquad \textbf{(b)} \begin{cases} 2x - 5z = -8 \\ 3y + 4z = 17 \\ x + y + z = 6 \end{cases} \qquad \textbf{(c)} \begin{cases} 5 + 2t = 4 - u \\ 1 - 5t = 2 + 3u \end{cases}$$

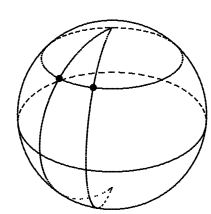
264. (Continuation) Calculate the following matrix products:

$$\textbf{(a)} \begin{bmatrix} 5 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1.5 \\ 2.5 \end{bmatrix} \quad \textbf{(b)} \begin{bmatrix} 2 & 0 & -5 \\ 0 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad \textbf{(c)} \begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} \quad \textbf{(d)} \begin{bmatrix} 2 & -5 & 4 \\ 6 & -1 & 3 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

What do your results tell you about the solutions to the systems in the preceding question?

265. A hemispherical bowl with a 30-centimeter radius contains some water, which is 12 centimeters deep. Find the volume of the water, to the nearest cubic centimeter.

266. A triangle has sides 13, 14, and 15. What is the radius of its circumscribed circle?



267. Salem, Oregon is 30 degrees due west of St. Paul, Minnesota, the latitude of both cities being 45° north of the equator. How far is it
(a) from Salem to the equator?
(b) from St. Paul to Salem, traveling due west along the circle of latitude?

268. If the equations $y = \sin x$ and $y = \cos x$ are both graphed on the same xy-axis system, the curves will intersect many times. Find coordinates for at least two intersection points.

269. Is it generally true that matrix products **MN** and **NM** are the same? Explain.

270. Find polar coordinates for the point described by $x = 4$ and $y = -7$.

271. A squash ball fits snugly inside a cubical box whose edges are 4 cm long. Guess the percentage of the box's volume that the ball occupies, and then calculate that percentage. (This is an example of a *sphere inscribed in a cube*.)

272. Find the volume of material that makes up the earth's crust, which is ten miles thick. Knowing this volume should make it fairly easy to estimate the surface area of the earth. (In fact, it is an especially simple calculation for members of the Flat Earth Society.) Your estimate is either larger or smaller than the exact area. Which? How do you know?

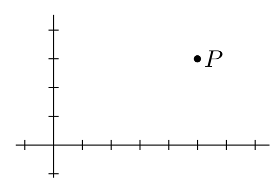
273. Calculate $\sin 72$ and $\sin(-72)$. Explain why $\sin(-\theta)$ is always the same as $-\sin\theta$. What can be said about $\cos(-\theta)$?

274. A system of simultaneous linear equations can always be written in matrix form as $\mathbf{CV} = \mathbf{B}$, where $\mathbf{C}$ is the matrix of *coefficients* and $\mathbf{V}$ is the matrix of (unknown) *variables*. Verify that $\mathbf{V} = \mathbf{C}^{-1}\mathbf{B}$ in fact satisfies this equation. It follows that

$$\begin{bmatrix} 2 & -5 & 4 \\ 6 & -1 & 3 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 12 \\ 3 \end{bmatrix} \quad \text{is solved by computing} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -5 & 4 \\ 6 & -1 & 3 \\ -5 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -7 \\ 12 \\ 3 \end{bmatrix}.$$

Use a calculator to finish the solution to this system of equations.

275. Apply each of the following transformations to $P = (5, 3)$:
(a) Reflect across the x-axis, then reflect across the line $y = x$.
(b) Reflect across the line $y = x$, then reflect across the x-axis.
(c) Rotate 90 degrees counterclockwise around the origin.



276. On the graph of $y = \sin x$, there are many points that have 0.39073 as the y-coordinate. Among these points, find the three that have the smallest positive x-coordinates.

277. On the graph of $y = \cos x$, many points have 0.39073 as their y-coordinate. Among them, find the three that have the smallest positive x-coordinates.

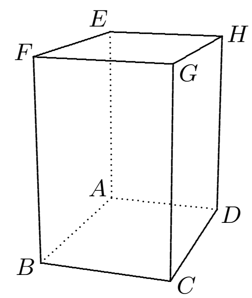
278. Describe the transformation whose matrix is given by $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}$.

279. A wheel of radius 5 is placed in the corner of the first quadrant of the xy-plane so that it is tangent to both axes. A paint mark is made on the wheel at the point where it touches the x-axis. What is the position of this paint mark after the wheel has rolled 8 units along the x-axis in the positive direction?

280. Use matrix methods to solve the following systems of linear equations:

(a) $\begin{cases} 3x + y = 11 \\ 4x + 3y = -7 \end{cases}$ **(b)** $\begin{cases} 5x + 2y + z = 9 \\ x - 4y - 3z = 1 \\ -3x + 4z = -2 \end{cases}$ **(c)** $\begin{cases} 1.69x + 9.61y - 2.56z = 19.36 \\ 3.61x - 7.84y + 1.21z = 11.56 \\ 4.41x - 5.29y - 6.76z = 12.25 \end{cases}$

281. A spider is on the rim of an empty conical cup when it spies a fly one third of the way around the rim. The cone is 36 cm in diameter and 24 cm deep. In a hurry for lunch, the spider chooses the shortest path to the fly. How long is this path?



282. In the rectangular framework shown at right, GAC is a 40-degree angle, CAB is a 33-degree angle, segments AB and AD lie on the x-axis and y-axis, respectively, and $AG = 10$. Find the coordinates of G.

283. Without using a calculator, choose **(a)** the larger of $\cos 40$ and $\cos 50$; **(b)** the larger of $\sin 40$ and $\sin 50$. Explain your reasoning.

284. As a spherical glob of ice cream that once had a 2-inch radius melts, it drips into a cone of the same radius. The melted ice cream exactly fills the cone. What is the height of the cone?

285. The circumference of a circle of latitude is two thirds of the circumference of the equator. What is the latitude?

286. How far from its center should a grapefruit with a 6-inch diameter be sliced, in order that both circular sections have the same radius as the two halves of a perfectly sliced orange with a 4-inch diameter?

287. Simplify the matrix product $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$.

288. (Continuation) Asked to find an expression that is equivalent to $\cos(\alpha + \beta)$, a student responded $\cos\alpha + \cos\beta$. What do you think of this answer, and why?

289. The earth rotates, so a person living in Quito (on the equator) travels 24881 miles in a day — nearly 1037 miles per hour. To the nearest mph, what rotational speed applies in Exeter, which is 43 degrees north of the equator? What rotational speed applies in your home town?

290. Given that $\sin\theta$ is $12/13$, with $90 < \theta < 180$, find the values of $\cos\theta$ and $\tan\theta$. Try to do this without finding θ first.

291. Given that $\sin\theta$ is a/b, with $0 < \theta < 90$, $a > 0$, and $b > 0$, find expressions for $\cos\theta$ and $\tan\theta$.

292. Faced with the problem of multiplying 5^6 times 5^3, Brook is having trouble deciding which of these four answers is correct: 5^{18}, 5^9, 25^{18}, or 25^9. Your assistance is needed. Once you have answered Brook's question, experiment with other examples of this type until you are able to formulate the *common-base principle for multiplication* of exponential expressions.

293. A 6×8 metal plate is resting inside a hemispherical bowl, whose radius is 13. The plate is parallel to the rim of the bowl, which is parallel to the tabletop on which the bowl is sitting. How far is it from the plate to the bottom of the bowl?

294. Exponents are routinely encountered in scientific work, where they help invesigators deal with large numbers:
(a) The human population of earth is nearly 7000000000, which is usually expressed in *scientific notation* as 7×10^9. The average number of hairs on a human head is 5×10^5. Use scientific notation to estimate the number of human head hairs on earth.
(b) Light moves very fast — approximately 3×10^8 meters every second. At that rate, how many meters does light travel in one year, which is about 3×10^7 seconds long? This so-called *light-year* is used in astronomy as a yardstick for measuring even greater distances.

295. What is the earth's rotational speed (in miles per hour) at a site whose latitude is θ degrees?

296. Find the volume of material that is needed to form a spherical shell whose outer radius is 6.0 inches and whose thickness is 0.01 inch. Use your answer to estimate the *surface area* of the 6-inch sphere.

297. An xyz-coordinate system is placed with its origin at the center of the earth, so that the equator (consisting of points with 0-degree latitude) is in the xy-plane, the North Pole (the only point with 90-degree latitude) has coordinates $(0, 0, 3960)$, and the *prime meridian* (see the next paragraph) is in the xz-plane. Where the prime meridian crosses the equator, the positive x-axis emerges from the South Atlantic Ocean, near the coast of Ghana.

The prime meridian is the *great semicircle* that runs through Greenwich, England on its way from the North Pole to the South Pole. Points on this meridian are all said to have *longitude* 0 degrees. The point $(0, 3960, 0)$ has longitude 90 degrees east, and the point $(0, -3960, 0)$ has longitude 90 degrees west. Thus the positive y-axis points east, into the Indian Ocean.
(a) Make a large diagram of this coordinate system.
(b) The latitude of Greenwich is approximately 51 degrees north. What are its xyz-coordinates?
(c) There is a point on the equator whose longitude is 33 degrees east. What are its xyz-coordinates?
(d) The latitude of Ankara, Turkey, is approximately 40 degrees north. What is its z-coordinate? The longitude of Ankara is approximately 33 degrees east. What are its xy-coordinates?

298. (Continuation) To the nearest mile, how far is it from Ankara to Greenwich? Travel along the surface of the earth, instead of tunneling through it.

299. Is it possible for $\sin\theta$ to be exactly twice the size of $\cos\theta$? If so, find such an angle θ. If not, explain why not.

300. There is a unique parabola that goes through the points $P = (-1, 1)$, $Q = (1, 1)$, and $R = (5, 6)$ that can be described by $y = ax^2 + bx + c$. Find coefficients a, b, and c.

301. The diameter and the slant height of a cone are both 24 cm. Find the radius of the largest sphere that can be placed inside the cone. (The sphere is therefore tangent to the base of the cone.) The sphere occupies a certain percentage of the cone's volume. First estimate this percentage, then calculate it.

302. Graph each pair of equations on a separate system of coordinate axes:
(a) $y = \sin x$ and $y = 2\sin x$ **(b)** $y = \sin x$ and $y = 3\sin x$ **(c)** $y = \sin x$ and $y = 0.5\sin x$
In general, what do the graphs of $y = a\sin x$ and $y = \sin x$ have in common, and how do they differ? What if the coefficient a is negative? The value of $|a|$ is called the *amplitude*.

303. Multiply the trinomial $x^2 + xy + y^2$ times the binomial $x - y$. The result is very simple.

304. Write an expression for the volume of the spherical shell formed between two concentric spheres, the inner one of radius r, the outer one of radius R. Factor your answer so that it has the form $\frac{4}{3}\pi \cdot (\textit{trinomial}) \cdot (\textit{binomial})$. In this situation, what is the meaning of the *binomial*? What can be said about the value of the *trinomial* when the *binomial* has a very small value? Make a conjecture concerning the *surface area* of a sphere of radius R.

305. Given that $\cos\theta$ is $7/25$, with $270 < \theta < 360$, find $\sin\theta$ and $\tan\theta$, without finding θ.

306. Without using a calculator, choose the larger of $\cos 310$ and $\cos 311$. Explain.

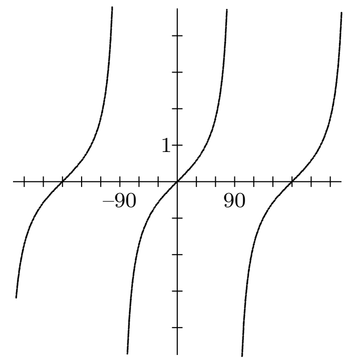
307. Describe the configuration of all points whose polar coordinate r is 3. Describe the configuration of all points whose polar coordinate θ is 110.

308. For each of the following pairs of matrices, calculate $\mathbf{MN}$ and $\mathbf{NM}$:

(a) $\mathbf{M} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ **(b)** $\mathbf{M} = \begin{bmatrix} 3 & -4 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$

(c) $\mathbf{M} = \begin{bmatrix} 1 & 2 \\ -6 & -3 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} -4 & 1 \\ -3 & -6 \end{bmatrix}$

309. (Continuation) Find two new matrices $\mathbf{M}$ and $\mathbf{N}$ that have the so-called *commutative property* $\mathbf{MN} = \mathbf{NM}$ that was noticed in part (c) of the preceding.



310. A graph of $y = \tan x$ is shown at right, drawn in degree mode. Confirm that one of the points on this graph is $(63.56, 2.011)$. Recall that the number 2.011 can be interpreted as the *slope* of a certain ray drawn from the origin. What ray? In contrast to a sine graph, which is a connected curve, this graph is in pieces. Explain why.

311. The equation $\tan\theta = 0.9004$ has infinitely many solutions. Find a way of describing all these values of θ.

312. Find the three smallest positive solutions to $2\sin\theta = -1.364$.

313. Explain why equation $\tan\theta = -2$ has solutions, but equation $\sin\theta = -2$ does not.

314. Without using a calculator, choose the larger of $\sin 76$ and $\sin 106$. Explain.

315. Faced with the problem of calculating $\left(5^4\right)^3$, Brook is having trouble deciding which of these three answers is correct: 5^{64}, 5^{12}, or 5^7. Once you have answered Brook's question, experiment with other examples of this type until you can formulate the principle that applies when exponential expressions are raised to powers.

316. The diameter of a typical atom is so small that it would take about 10^8 of them, arranged in a line, to reach just one centimeter. It is therefore a plausible estimate that a cubic centimeter could contain about $10^8 \times 10^8 \times 10^8 = \left(10^8\right)^3$ atoms. Write this huge number as a power of 10.

317. Reflect the graph $y = 2\sin x$ across the x-axis. Find an equation to describe the curve that results. Use a graphing tool to check your answer.

318. Avery is riding a Ferris wheel that turns once every 24 seconds, and whose radius is 8 meters. The function h defined by $h(t) = 9 - 8\cos(15t)$ describes Avery's distance from the ground (in meters) after t seconds of riding. For example, $h(8) = 13$ means that Avery is 13 meters above the ground after 8 seconds of riding. By the way, "h of 8" or "h at 8" are two common ways to say $h(8)$.
(a) Evaluate $h(0)$, and explain its significance.
(b) Explain why $h(16) = h(8)$.
(c) Find a value for t that fits the equation $h(t) = 10$. Interpret this t-value in the story.
(d) Explain why $h(t + 24) = h(t)$ is true, no matter what value t has.
(e) What is the complete range of values that $h(t)$ can have?
It is not unusual to see the equation $h = 9 - 8\cos 15t$ used to define the function h, even though h is written instead of $h(t)$ and the parentheses around $15t$ are missing.

319. At any given moment, the *terminator* is the circle on the earth's surface that separates night from day. Is the terminator a great circle? Explain.

320. Given vectors $\mathbf{u} = [5, 12]$ and $\mathbf{v} = [4, 3]$, Remy was asked to evaluate $\dfrac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{u} \bullet \mathbf{u}}$. Remy simply calculated $\dfrac{13 \cdot 5}{13 \cdot 13} = \dfrac{5}{13}$. What do you think of this calculation? What is the meaning of the ratio Remy was asked to calculate?

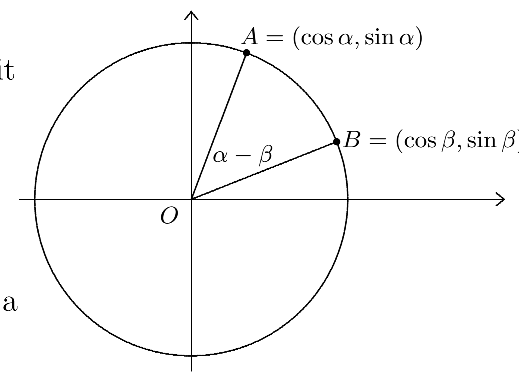
321. An xyz-coordinate system is placed with its origin at the center of the earth, so that the equator is in the xy-plane, the North Pole has coordinates $(0, 0, 3960)$, and the xz-plane contains the prime meridian, which is the great semicircle that runs through Greenwich, England on its way from the North Pole to the South Pole. Recall that the y-axis is oriented so that it points east, into the Indian Ocean. Find the coordinates (x, y, z) of Exeter NH, whose latitude and longitude are 43 degrees north and 71 degrees west.

322. The lanes on the circular outdoor track are one meter wide. By running in the lane next to the innermost lane, Corey ran extra distance in an eight-lap race. How much?

323. Find the volume of the largest cube that fits in a sphere of radius 8 cm.

324. Describe the configuration of all points on the earth's surface
(a) whose latitude is 43° N;
(b) whose longitude is 71° W.

325. In the diagram, points A and B are on the unit circle, and O is the origin.
(a) Explain why the central angle BOA is labeled $\alpha - \beta$.
(b) Obtain a formula for $\cos(\alpha - \beta)$ by applying the Law of Cosines in its vector form.
(c) By replacing β by $-\beta$ in your formula, obtain a familiar formula for $\cos(\alpha + \beta)$.



326. Let $A = (-6, -4)$, $B = (3, 2)$, and $C = (6, 4)$.
(a) These points lie on a line through the origin. Find its slope.
(b) Let $\mathbf{u}$ be the vector whose components are the x-coordinates of A, B, and C, and let $\mathbf{v}$ be the vector whose components are the y-coordinates of A, B, and C. Show that $\mathbf{v}$ is a positive scalar multiple of $\mathbf{u}$ (thus $\mathbf{u}$ and $\mathbf{v}$ point in the same direction).
(c) Explain why the scalar multiple in part (b) equals the slope you found in part (a).
(d) What would the vectors $\mathbf{u}$ and $\mathbf{v}$ have looked like if A, B, and C had not been collinear with the origin?

327. There is more than one triangle PQR that can be described using the data $p = 13$, $r = 14$, $\sin P = 4/5$. For each triangle, find q (the length of the third side), the sizes of the angles, and make a sketch.

328. Given that $\sin\theta = k$, and that $90 < \theta < 180$, find expressions for $\cos\theta$ and $\tan\theta$.

329. On the sphere $x^2+y^2+z^2 = 13^2$, there are many great circles that intersect at $(3, 4, -12)$. Find coordinates for the *other* point where these circles all intersect.

330. A spherical globe, 12 inches in diameter, is filled with spherical gumballs, each having a 1-inch diameter. Estimate the number of gumballs in the globe, and explain your reasoning.

331. Convert the polar pair $(r = 8; \theta = 150)$ to an equivalent Cartesian pair (x, y).

332. Let $A = (-3, -2)$, $B = (-1, -1)$, $C = (4, 3)$, $\mathbf{u} = [-3, -1, 4]$, and $\mathbf{v} = [-2, -1, 3]$.
(a) Show that $\mathbf{u}$ and $\mathbf{v}$ point in different directions. Let $\mathbf{w}$ be the vector that results when $\mathbf{v}$ is projected onto $\mathbf{u}$. Show that $\mathbf{w}$ is approximately $[-2.19, -0.73, 2.92]$.
(b) Make a scatter plot. Verify that A, B, and C are not collinear. Notice that the x-coordinates of these points are the components of $\mathbf{u}$ and the y-coordinates are the components of $\mathbf{v}$.
(c) Verify that the points $A' = (-3, -2.19)$, $B' = (-1, -0.73)$, and $C' = (4, 2.92)$ lie on a line that goes through the origin. Notice that the y-coordinates of A', B', and C' appear as the components of $\mathbf{w}$, and that they are proportional to the components of $\mathbf{u}$.

333. (Continuation)
(a) You calculated $\mathbf{w}$ by first finding that it is m times as long as $\mathbf{u}$, where m is $\dfrac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{u}\bullet\mathbf{u}}$. Notice that m is also the slope of the line through A', B', and C'. Now use a calculator to find an equation for the so-called *regression line* (or LinReg) for the data points A, B, and C. The slope should look familiar.
(b) Verify that the vector $\mathbf{r} = \mathbf{v} - \mathbf{w}$ is perpendicular to $\mathbf{u}$, then explain why this should have been expected. It is customary to call $\mathbf{r}$ a *residual vector*, because it is really just a list of residuals. Review the meaning of this data-analysis term if you need to.
(c) The regression line is sometimes called the *least-squares* line of best fit, because $\mathbf{w}$ was chosen to make $\mathbf{r}$ as short as possible. Explain this terminology. You will need to refer to the Pythagorean formula for calculating the length of a vector.

334. To the nearest square mile, what is the surface area of the earth? How much area is found between the meridians 40 degrees west and 75 degrees west?

335. Faced with the problem of dividing 5^{24} by 5^8, Brook is having trouble deciding which of these four answers is correct: 5^{16}, 5^3, 1^{16}, or 1^3. Your assistance is needed. Once you have answered Brook's question, experiment with other examples of this type until you can formulate the *common-base principle for division* of exponential expressions. Then apply this principle to the following situations:
(a) earth's human population is roughly 6×10^9, and its total land area excluding the polar caps is roughly 5×10^7 square miles. If the human population were distributed uniformly over all available land, approximately how many persons would be found per square mile?
(b) At the speed of light, which is 3×10^8 meters per second, how many seconds does it take for the sun's light to travel the 1.5×10^{11} meters to earth?

336. The area of a circle of radius r is πr^2, which happens to be $\frac{1}{2}\,r$ times the circumference $2\pi r$. Explain why this relationship should be expected. One way is to apply your knowledge of circular sectors. Another way is to consider a billion-sided regular polygon that is circumscribed around a circle of radius r; how are its area and perimeter related?

337. (Continuation) The volume enclosed by a sphere of radius r is $\frac{4}{3}\,\pi r^3$. The surface area of the same sphere is $4\pi r^2$. You may already have noticed that the volume is exactly $\frac{1}{3}\,r$ times the surface area. Explain why this relationship should be expected. One way is to consider a billion-faceted polyhedron that is circumscribed about a sphere of radius r; how are its volume and surface area related?

338. Graph each pair of functions on a separate system of coordinate axes:
(a) $y = \sin x$ and $y = \sin 2x$ **(b)** $y = \sin x$ and $y = \sin 3x$ **(c)** $y = \sin x$ and $y = \sin 0.5x$
What do the graphs of $y = \sin mx$ and $y = \sin x$ have in common, and how do they differ?

339. Tickets for a concert were sold in three categories: adult, child, and senior citizen. For each type, the number of tickets sold for the three performances is shown in the matrix. The box office receipts were \$2715 for Friday, \$2613 for Saturday, and \$2412 for Sunday. Find the cost of each type of ticket.

$$\begin{array}{cc} & \begin{array}{ccc} \textit{Child} & \textit{Adult} & \textit{Senior} \end{array} \\ \begin{array}{c} Fri \\ Sat \\ Sun \end{array} & \begin{bmatrix} 35 & 120 & 15 \\ 22 & 124 & 12 \\ 58 & 96 & 6 \end{bmatrix} \end{array}$$

340. Use a calculator to find the equation of the least-squares line (LinReg) for the five data points (2.0,3.2), (3.0,3.5), (5.0,5.0), (7.0,5.8), and (8.0,6.0). Let G be the *centroid* of these points — its x-coordinate is the average of the five given x-coordinates, and its y-coordinate is the average of the five given y-coordinates. Verify that G is on the least-squares line.

341. Explain why the value of $[\cos\theta, \sin\theta] \bullet [\cos(90+\theta), \sin(90+\theta)]$ is independent of θ.

342. The common-base principle for multiplication predicts that $5^{1/2}$ times $5^{1/2}$ should be 5. Explain this logic, then conclude that $5^{1/2}$ is just another name for a familiar number. Use a calculator to check your prediction. How would you describe the number $6^{1/3}$, given that $6^{1/3} \cdot 6^{1/3} \cdot 6^{1/3}$ equals 6? Formulate a general meaning of expressions like $b^{1/n}$, and use a calculator to test your interpretation on simple examples like $8^{1/3}$ and $32^{1/5}$.

343. Water is being poured into a conical cup that is 12 cm tall.
(a) When the water in the cup is 9 cm deep, what percentage of the cup is filled?
(b) When the cup is 75 percent filled, how deep is the water?

344. Plot the four data points $(-3,-2)$, $(-2,1)$, $(1,0)$, and $(4,1)$.
(a) Verify that the centroid of the data is at the origin. Draw a line through the origin that looks like it does a good job of fitting this data. Let m be its slope.
(b) For each of the four points (x,y), the *residual* is $y-mx$, which depends on the variable slope m. For example, the residual for the first point is $-2-(-3m)$, or just $3m-2$. Calculate the other three residuals, then square each of the four residuals and simplify the *sum* of the four squares. The result should be a quadratic polynomial, in which m is the variable.
(c) The method of *least squares* seeks the m-value that *minimizes* this sum of squared residuals. Find this value of m. If you use a calculator, you may need to use the symbol x in place of m. Compare your m-value with the slope of the line you drew in part (a).
(d) Confirm your answer by using a calculator's least-squares (LinReg) capability.

345. The matrix $\mathbf{M} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$ defines an isometry of the xy-plane.
(a) What special properties do the column vectors of this matrix have?
(b) Verify that the point $(2,4)$ remains stationary when $\mathbf{M}$ is applied to it.
(c) What is the significance of the stationary point $(2,4)$? What does it tell you about the possible isometries that $\mathbf{M}$ could be? Do other points invite examination?
(d) Show that $\mathbf{MM}$ is the 2×2 identity matrix. What does this suggest about the geometric transformation that $\mathbf{M}$ represents? Confirm your suspicions.

346. Graph each of the following pairs of functions on a separate system of coordinate axes. What does each pair of graphs have in common? How do the graphs differ?
(a) $y = 2\cos x$ and $y = 1 + 2\cos x$ **(b)** $y = -3\cos x$ and $y = 1 - 3\cos x$

347. For each of the following, there are two points on the unit circle that fit the given description. Without finding θ, describe how the two points are related to each other.
(a) $\cos\theta = -0.4540$ **(b)** $\sin\theta = 0.6820$ **(c)** $\tan\theta = -1.280$

348. A cylinder of radius 4 and height h is inscribed in a sphere of radius 8. Find h.

349. A cone with 10-inch diameter is 12 inches tall.
(a) Find the radius of the largest sphere that can be inscribed in the cone.
(b) The volume of this sphere is what percentage of the volume of the cone?

350. The table at right shows how many seconds are needed for a stone to fall to earth from various heights (measured in meters). Make a scatter plot of this data. Explain how the data suggests that the underlying relationship is not linear.

(a) Calculate the squares of the times and enter them in a third column. A scatter plot of the relation between the first and third columns does suggest a linear relationship. Use LinReg to find it, letting x stand for height and y stand for the square of the time.

(b) It is now easy to write a nonlinear relation between h and t by expressing t^2 in terms of h. Use this equation to predict how long it will take for a stone to fall from a height of 300 meters.

height	*time*
10	1.42
20	2.01
30	2.46
40	2.85
50	3.18
60	3.49
70	3.76
80	4.02
90	4.27
100	4.50

351. The sides of a triangle are 10, 17, and 21 inches long. Find
(a) the smallest angle of the triangle; **(b)** the diameter of the circumscribed circle.

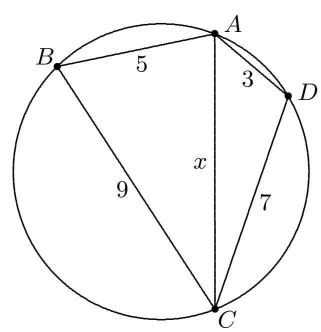
352. A clear plastic cone is 9 inches tall, with some liquid sealed inside. When the cone is held point down, the liquid is 8 inches deep. When the cone is inverted and held point up, the liquid is d inches deep. Find d, to the nearest hundredth of an inch.

353. The result of dividing 5^7 by 5^3 is 5^4. What is the result of dividing 5^3 by 5^7, however? By considering such examples, decide what it means to put a *negative* exponent on a base.

354. Exponents are routinely encountered in science, where they help to deal with small numbers. For example, the diameter of a proton is 0.0000000000003 cm. Explain why it is logical to express this number in scientific notation as 3×10^{-13}. Calculate the surface area and the volume of a proton.

355. Graph the equation $y = 1 + 2\sin x$. This curve crosses the x-axis in several places. Identify all the x-intercepts with $0 < x < 360$.

356. Quadrilateral $ABCD$ is inscribed in a circle, and the lengths of its sides are $AB = 5$, $BC = 9$, $CD = 7$, and $DA = 3$. Let x be the unknown length of diagonal AC.



(a) In terms of x, write an expression for $\cos B$.

(b) In terms of x, write an expression for $\cos D$.

(c) A simple relationship holds between angles B and D. Use it to help you find the unknown length x.

(d) Find the length of diagonal BD.

357. The value of $[\cos\theta, \sin\theta] \bullet [\cos(180+\theta), \sin(180+\theta)]$ does not depend on the value of θ. Explain why.

358. Given polar coordinates r and θ for a point, how do you calculate the Cartesian coordinates x and y for the same point?

359. An xyz-coordinate system is placed with its origin at the center of the earth, so that the equator is in the xy-plane, the North Pole has coordinates $(0, 0, 3960)$, and the xz-plane contains the prime meridian. Find the coordinates (x, y, z) of Osaka, Japan, whose latitude and longitude are 34.7 degrees north and 135.5 degrees east.

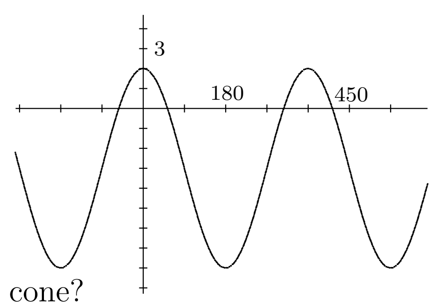
360. The results of three PEA dorm outings to Burger Palace: Dutch spent \$54.39 for 14 burgers, 12 shakes, and 15 fries; Wentworth spent \$291.95 for 81 burgers, 62 shakes, and 72 fries; Lamont spent \$111.93 for 25 burgers, 33 shakes, and 29 fries. How much does an order of fries cost at Burger Palace? It may be helpful to label the rows and columns of your matrices.

361. Matrices $\mathbf{M} = \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}$ represent reflections in the lines $y = 2x$ and $3y = x$, respectively. Verify that $\mathbf{MN}$ is not equal to $\mathbf{NM}$, and explain why this should have been expected. What transformations do the two products represent?

362. Centered 7 meters above the ground, a Ferris wheel of radius 6 meters is rotating with angular speed 24 degrees per second. Assuming that Harley's joyride began at time $t = 0$ seconds at the lowest point on the wheel, write a formula for the function that describes the distance $h(t)$ from Harley to the ground (in meters) after t seconds of riding.

363. (Continuation) Draw a graph of $h(t)$ for the restricted *domain* $0 \le t \le 30$, and find coordinates for two points on your graph that both represent the situation when Harley is 10 meters above the ground and climbing. Interpret the domain restriction in context.

364. As shown, the graph $y = -3 + 5\cos x$ intersects the x-axis repeatedly. Find all the x-intercepts that appear in the illustration.



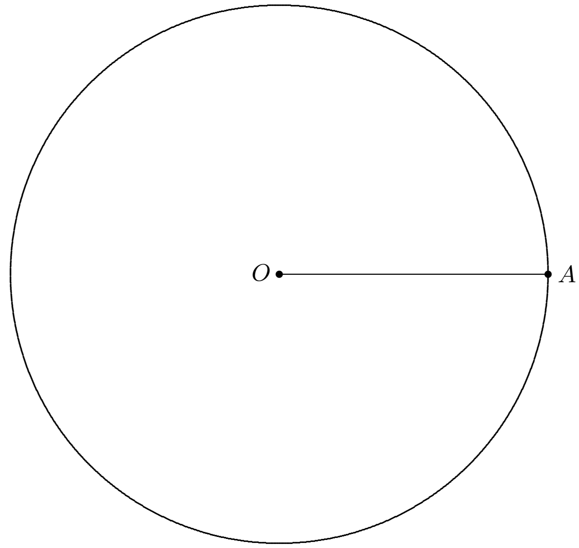
365. Both the slant height and the base diameter of a cone are 12 inches. How far is it from a point on the base circle to the diametrically opposite point on the circle, if it is required that the path must lie on the lateral surface of the cone?

366. A spherical ball weighs three times as much as another ball of identical appearance and composition. The second ball weighs less because it is actually hollow inside. Find the radius of the hollow cavity in the second ball, given that each ball has a 5-inch radius.

367. You have used matrices to calculate the results of certain rotations and reflections. Which ones? Are translations calculated using matrices?

368. It is well known that multiplication can be *distributed* over addition or subtraction, meaning that $a \cdot (b + c)$ is equivalent to $a \cdot b + a \cdot c$, and that $a \cdot (b - c)$ is equivalent to $a \cdot b - a \cdot c$. It is *not* true that multiplication distributes over multiplication, however, for $a \cdot (b \cdot c)$ is not the same as $a \cdot b \cdot a \cdot c$. Now consider distributive questions about exponents: Is $(b + c)^n$ equivalent to $b^n + c^n$? Explore this question by choosing some numerical examples. Is $(b \cdot c)^n$ equivalent to $b^n \cdot c^n$? Look at more examples.

369. The circle shown at right is centered at O. Use a licorice strip to find a point B on this circle for which minor arc AB has the *same length* as the radius OA. Draw radius OB and use a protractor to measure the size of angle AOB. Your answer should be close to 60 degrees. By considering triangle AOB and the relation between the arc AB and its chord, explain why angle AOB must in fact be smaller than 60 degrees.



Angle AOB is an example of a *radian* — a central angle whose arc has the same length as the radius of the circle in which it is drawn.

370. (Continuation) How many 1-radian arcs does it take to fill a complete circle? First make an estimate using the licorice-strip approach, then look for a theoretically exact answer. Do any of your answers depend on the size of the circle used?

371. A 6-inch arc is drawn using a 4-inch radius. Describe the angular size of the arc **(a)** using radians; **(b)** using degrees.

372. A 2.5-radian arc is drawn using a radius of 6 inches. How long is the arc?

373. Find equivalent ways to rewrite (without using a calculator) the following expressions:

(a) $\dfrac{6a^8}{3a^4}$ **(b)** $(3p^3q^4)^2$ **(c)** $b^{1/2}b^{1/3}b^{1/6}$ **(d)** $\left(\dfrac{2x^3}{3y^2}\right)^2$ **(e)** $\left(d^{1/2}\right)^6$

374. Given the points $A = (4, 7, 1)$, $B = (12, -1, 5)$, and $C = (2, -3, 12)$, find the area of triangle ABC.

375. Invent a division problem whose answer is b^0, and thereby discover the meaning of b^0.

376. Let $\mathbf{J}$ be the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and calculate $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$. Do you detect a pattern?

377. (Continuation) Viewed as a geometric transformation, what effect does $\mathbf{J}$ have? To find out, apply $\mathbf{J}$ to a variety of points, including those in the *unit square*. Is the image of the unit square another square?

378. Given that $\tan\theta = 2.4$, with $180 < \theta < 270$, without a calculator, find the exact values of $\sin\theta$ and $\cos\theta$. Are your answers *rational numbers*?

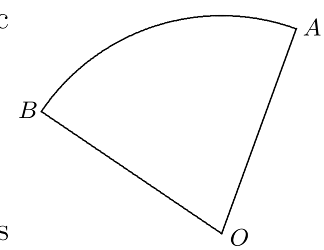
379. Show that the surface area of a sphere is two thirds of the total surface area of any cylinder circumscribed around the sphere (the sphere therefore touches both bases of the cylinder as well as its lateral surface).

380. With a graphing tool in degree mode, using the window $0 \le x \le 360$ and $-2 \le y \le 2$, examine the graph of $y = \sin x$. Explain why its first positive x-intercept has the value it does. Then graph the same equation *in radian mode*. The display will show many more oscillations than before, making it difficult to see the first positive x-intercept. To make that intercept clearer, reduce the window to $0 \le x \le 10$ and $-2 \le y \le 2$, then obtain an accurate reading of this x-value. Once you recognize it, explain how its value could have been predicted.

381. Graph each pair of equations on a separate system of coordinate axes:
(a) $y = \cos x$ and $y = \cos 2x$ **(b)** $y = \cos x$ and $y = \cos 3x$ **(c)** $y = \cos x$ and $y = \cos 0.5x$
What do the graphs of $y = \cos x$ and $y = \cos mx$ have in common, and how do they differ?

382. You have deposited \$1000 in a money-market account that earns 8 percent annual interest. Assuming no withdrawals or additional deposits are made, calculate how much money will be in the account one year later; two years later; three years later; t years later.

383. Give two reasons why the projection of $\mathbf{u}$ onto $\mathbf{v}$ is not the same as the projection of $\mathbf{v}$ onto $\mathbf{u}$.



384. Find the size of central angle AOB, given that the length of arc AB is 16 cm and the length of radius OA is 12 cm.

385. (Continuation) Find the size of angle AOB, given that the length of arc AB is 8 cm and the length of radius OA is 6 cm.

386. (Continuation) To find the size of the central angle AOB, it is enough to know the value of what ratio?

387. The radius of a circle is 9, and arc PQ has length 22. Find the length of chord PQ.

388. (Continuation) Working in *radian* mode, evaluate $2 \cdot 9 \cdot \sin(11/9)$. Notice that this expression provides the correct answer to the chord-length question. Why is this so? In particular, what angle does the number $11/9$ describe?

389. Consider a circle centered at O, and one of its arcs PQ. To two decimal places, find the *radian measure* of angle POQ, when its degree measure is
(a) 45 **(b)** 75 **(c)** 100 **(d)** 180 **(e)** 360

390. You have seen how rotations and reflections can be represented using matrices, but most 2×2 matrices do not represent isometries. Consider the matrix $\mathbf{K} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, for example. Apply $\mathbf{K}$ to the unit square. What is the image polygon and what is its area?

391. (Continuation) Let $\mathcal{R}$ be the region that results when the matrix $\mathbf{M} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is applied to the unit square. Show that the area of $\mathcal{R}$ is $|ad - bc|$. The number $ad - bc$ is called the *determinant* of the matrix $\mathbf{M}$.

392. When working in degree mode, we say that the *period* of the graph $y = \sin x$ is 360. What does this statement mean? What is the period of the graph of $y = \cos x$? What is the period of the graph $y = \tan x$? What is the period of the graph $y = \sin 2x$? What is the period of the graph $y = \sin mx$?

393. When working in radian mode, we say that the *period* of the graph $y = \sin x$ is 2π. What does this statement mean? What is the period of the graph of $y = \cos x$? What is the period of the graph $y = \tan x$? What is the period of the graph $y = \sin 2x$? What is the period of the graph $y = \sin mx$?

394. Explain your opinions of each of the following student responses:
(a) Asked to find an expression equivalent to $x^8 - x^5$, a student responded x^3.
(b) Asked to find an expression equivalent to $\dfrac{x^8 - x^5}{x^2}$, a student responded $x^6 - x^3$.
(c) Another student said that $\dfrac{x^2}{x^8 - x^5}$ is equivalent to $\dfrac{1}{x^6} - \dfrac{1}{x^3}$.

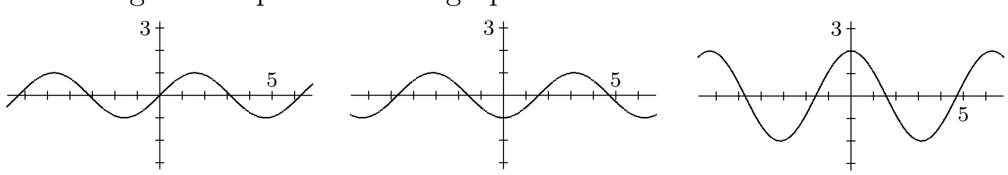
395. The table at right shows how many seconds are needed for a stone to fall to earth from various heights (measured in meters). You have already made a scatter plot of this nonlinear data. This time, calculate the *square roots* of the heights and enter them in a new column. A scatter plot of the relation between the second column and the new column should reveal a linear relationship. Find it, then use it to *extrapolate* how much time is needed for a stone to fall 300 meters.

height	*time*
10	1.42
20	2.01
30	2.46
40	2.85
50	3.18
60	3.49
70	3.76
80	4.02
90	4.27
100	4.50

396. Graph each of the following pairs of functions on a separate system of coordinate axes, and account for what you see:
(a) $y = \sin x$ and $y = \sin(-x)$ **(b)** $y = \cos x$ and $y = \cos(-x)$

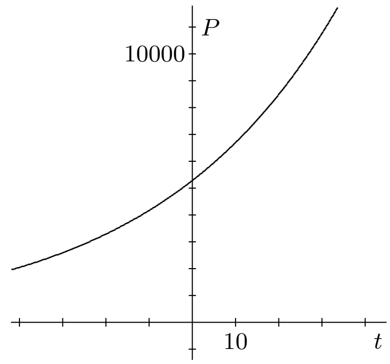
397. Find a way of describing all the intersections of the line $y = 1.5399$ and the graph of $y = \tan x$. For what numbers m is it possible to solve the equation $m = \tan x$?

398. The *sinusoidal* graphs shown below appeared on a calculator in radian mode. Find equations that might have produced the graphs.



399. The population of Grand Fenwick has been increasing at the rate of 2.4 percent per year. It has just reached 5280 (a milestone). What will the population be after ten years? after t years? After how many years will the population be 10 560?

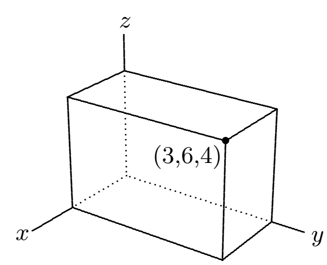
400. The diagram at right shows part of the graph of $P(t) = 5280(1.024)^t$, a function that describes a small town whose population has been growing at an annual rate of 2.4 percent.
(a) What is $P(0)$, and what is its meaning?
(b) Use the graph to estimate the solution of the equation $P(t) = 10\,560$.
(c) Calculate $P(-30)$. What does this number mean?
(d) Comment on the part of the graph that lies outside the borders of the illustration. How would it look if you could see it, and what does it mean?



401. Rewrite each equation so that it has the form "$x = \ldots$" Please do not use "solve."
(a) $x^5 = a^3$ **(b)** $x^{1/5} = a^3$ **(c)** $(1 + x)^{15.6} = 2.0$ **(d)** $x^{-2} = a$

402. In a circle of radius 5 cm, how long is a 1-radian arc? How long is a 2.2-radian arc?

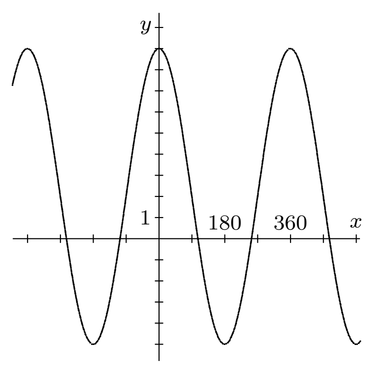
403. On a number line (a one-dimensional context), the equation $x = 3$ describes a single point. In a two-dimensional context, what does the equation $x = 3$ describe? What does the equation $x = 3$ describe in a three-dimensional context?



404. The illustration at right shows a rectangular box, three of whose edges coincide with the coordinate axes, and one of whose vertices is $(3, 6, 4)$. Give an equation for the plane containing the front face of the box. Do the same for the top and right faces.

405. Show that $P + Pr + (P + Pr)r = P(1 + r)^2$. Based on your work with exponential growth, interpret the three individual terms on the left side of this equation, and explain why their sum should equal the expression on the right side.

406. The equation whose graph is shown at right has the form $y = k + a\cos x$. Working in degree mode, find believable values for the coefficients a and k, and explain how these numbers affect the appearance of the graph.



407. An object moves around $x^2 + y^2 = 25$ (which represents a circle whose radius is 5 meters) at a constant speed. At time $t = 0$ seconds, the object is at $(5, 0)$. When $t = 1$, it is at $(4, 3)$. Where is the object when $t = 2$? when $t = 3$? when $t = n$? What is the object's speed? At what times does the object return to $(5, 0)$? To arrive at your answers, what assumptions did you make? Did your classmates make the same assumptions?

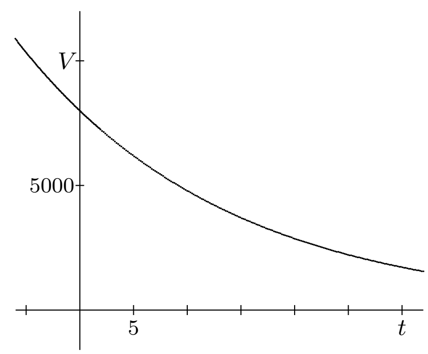
408. A helium-filled balloon is slowly deflating. During any 24-hour period, it loses 5 percent of the helium it had at the beginning of that period. The balloon held 8000 cc of helium at noon on Monday. How much helium did it contain 3 days later? 4.5 days later? 20 days later? n days later? 12 hours later? k hours later? Approximately how much time is needed for the balloon to lose half its helium? This time is called the *half-life*. Be as accurate as you can.

409. It is possible to fit more than forty superballs with 1-inch diameters into a rectangular box that is 8 inches long, 5 inches wide, and 1 inch tall. How many will the box hold? When the box is full and someone shakes it, will the superballs be free to move around?

410. Plot the points where the graph of $x+2y+3z = 27$ intersects the three coordinate axes. Also plot the points $(0, y, z)$ in the yz-plane that fit the equation. Then plot one point, all of whose coordinates are positive, that fits the equation. What geometric object is described by the equation $x + 2y + 3z = 27$?

411. Describe the configuration of points equidistant from $A = (2, 4, 1)$ and $B = (4, 8, 7)$. Write and simplify an equation that says that $P = (x, y, z)$ is equidistant from A and B.

412. The picture at right shows part of the graph of $V(t) = 8000(0.95)^t$. This function tells the story of a shrinking balloon that loses 5 percent of its helium each day.
(a) What is $V(0)$, and what is its significance?
(b) Use the graph to estimate the t-value that solves the equation $V(t) = 4000$.
(c) Calculate $V(-3)$. What does this value mean?
(d) Comment on the part of the graph that lies outside the borders of the illustration. How would it look if you could see it, and what does it mean?



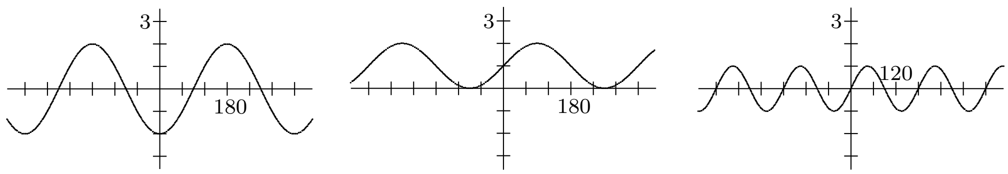
413. An xyz-coordinate system is placed with its origin at the center of the earth, so that the equator is in the xy-plane, the North Pole has coordinates $(0, 0, 3960)$, and the xz-plane contains the prime meridian. Find the coordinates (x, y, z) of Lusaka, Zambia, whose latitude and longitude are 15.3 degrees south and 28.5 degrees east.

414. The table at right contains experimental data. Each entry in the d-column stands for the distance from a photoelectric cell to a light source, and each entry in the E-column displays the amount of energy falling on the cell. A scientist suspects that the energy is closely related to the square of the distance. Create two new columns of transformed data, by calculating the *squares* of the d-entries and the *reciprocals* of the E-entries. Use the linear relationship between these variables to write down a simple equation that describes this example of an *inverse-square law*. Then predict the E-value that should correspond to a d-measurement of 36. Compare your equation with the theoretical prediction $E = 1444d^{-2}$.

d	E
12.0	10.1
15.0	6.41
18.0	4.48
21.0	3.30
24.0	2.52
27.0	2.01

415. Convert the following to equivalent forms in which no negative exponents appear:
(a) $\left(\frac{2}{5}\right)^{-1}$ **(b)** $\frac{6}{x^{-2}}$ **(c)** $\left(\frac{-3}{2}\right)^{-3}$ **(d)** $\frac{6xy}{3x^{-1}y^{-2}}$ **(e)** $\left(\frac{2x^2}{3x^{-1}}\right)^{-2}$

416. Working in degree mode, find plausible equations for each of the sinusoidal graphs below:



417. In order that a \$10000 investment grow to \$20000 in seven years, what must be the annual rate of interest? Seven years could be called the *doubling time* for this investment. Notice that it is being assumed that the interest is *compounded*.

418. Express the radius r of a sphere as a function of the volume V it encloses.

419. On one system of coordinate axes, graph the equations $y = 3^x$, $y = 2^x$, $y = 1.024^x$, and $y = \left(\frac{1}{2}\right)^x$. What do graphs of the form $y = b^x$ have in common? How do they differ?

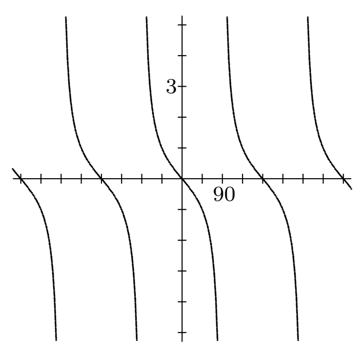
420. Make up a context for the expression $4000(1.005)^{12}$, in which the "12" counts months. In this context, what do the expressions $4000\left((1.005)^{12}\right)^n$ and $4000(1.0617)^n$ mean?

421. Find coordinates for two points that belong to the plane $2x + 3y + 5z = 15$, trying to choose points that no one else in the class will think of. Show that the vector $[2, 3, 5]$ is perpendicular to the segment that joins your two points.

422. (Continuation for class discussion) Explain why $[2, 3, 5]$ is perpendicular to the plane.

423. Suppose that (x_1, y_1, z_1) and (x_2, y_2, z_2) are points that fit the equation $ax + by + cz = d$. In other words, suppose that $ax_1 + by_1 + cz_1 = d$ and $ax_2 + by_2 + cz_2 = d$ are both true. Subtract one of these equations from the other, and interpret the result.

424. The graph at right appeared on a calculator in degree mode. Find an equation that might have produced this graph.

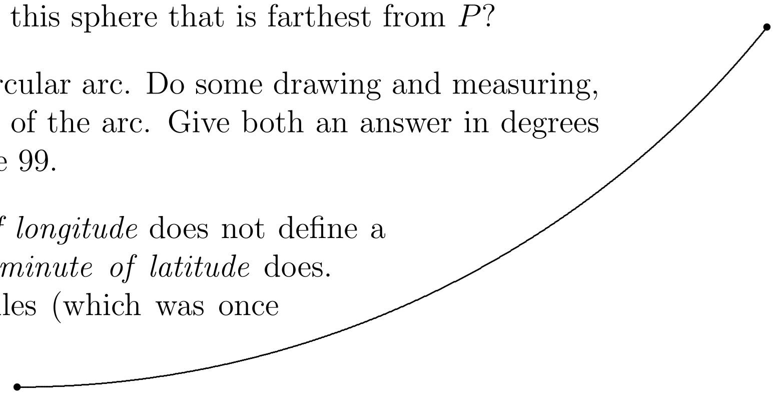


425. *Singular matrices.* Not all matrices have inverses. For example, consider $\mathbf{L} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Show that there is no matrix $\mathbf{N}$ for which $\mathbf{NL}$ is the 2×2 identity matrix. One way to proceed is to consider $\mathbf{L}$ as a transformation. What is the result of applying $\mathbf{L}$ to the unit square?

426. (Continuation) Invent another example of a *non-invertible* (singular) 2×2 matrix, in which the four entries are all different and nonzero.

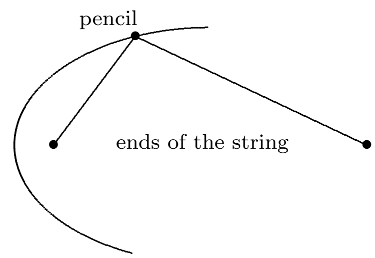
427. The point $P = (11, 2, 5)$ is on the sphere $x^2 + y^2 + z^2 - 10x + 2y - 14z = -26$. What are the coordinates of the point on this sphere that is farthest from P?

428. The figure at right shows a circular arc. Do some drawing and measuring, and then estimate the angular size of the arc. Give both an answer in degrees and an answer in radians. See page 99.



429. The expression *one minute of longitude* does not define a distance, but the expression *one minute of latitude* does. Explain. Find this distance in miles (which was once called a *nautical mile*).

430. *Kepler's First Law*: Planets follow *elliptical* orbits around the sun. As the following exercise shows, these curves are typically not circles, but are very much *like* circles. You need to find a piece of string that is approximately ten inches long, a large piece of paper, a sharp pencil, and a partner. Mark two points about 6 inches apart near the center of the paper. While your partner anchors the ends of the string on the marks, you use the pencil point to pull the string taut. Keep the string taut as you drag the pencil on a tour that surrounds the ends of the string. The resulting curve is called an *ellipse*. (This and the next exercise are well-suited to the classroom, where board space is also available.)



431. (Continuation) Let your partner do the drawing now. Try anchoring the ends of the string closer together or farther apart, and notice the effect that this has on the shape of the finished curve. To obtain an actual circle, how should the ends of the string be placed?

432. (Continuation) Imagine using graph paper for the preceding construction. Suppose that the string is 10 units long, and that its ends are held at $F_1 = (-3, 0)$ and $F_2 = (3, 0)$. Calculate the four axis intercepts of the ellipse. You should notice that there is a simple relationship between the length of the string and the distance between the x-intercepts.

433. The table at right contains astronomical data for the eight planets of our solar system. Each entry in the d-column stands for the *mean distance* from the planet to the sun, measured using the earth-to-sun mean distance (93 million miles) as the *astronomical unit.* Each entry in the t-column stands for the amount of time needed for the planet to make one revolution of its orbit, measured using earth years. Verify that this is non-linear data. Johannes Kepler, after working with such data for many years, empirically discovered in 1626 the relationship you are about to confirm. Create two new columns of transformed data, putting the *square roots* of the d-entries in one column and the *cube roots* of the t-entries in the other. Use the simple linear relationship between these variables to write an equation that expresses *Kepler's Third Law.*

d	t
0.387	0.241
0.723	0.615
1.000	1.000
1.524	1.881
5.203	11.86
9.539	29.46
19.19	84.07
30.06	164.8

(a) Calculate the length of time needed for the asteroid Ceres to orbit the sun, given that Ceres is 251 million miles from the sun. This is an *interpolation* question, by the way.
(b) If there were a planet that took 50 years to complete its orbit, how far from the sun would this planet have to be?

434. (Continuation) Kepler's Third Law can be written in a variety of equivalent forms. To see this, solve the equation **(a)** for d in terms of t, and **(b)** for t in terms of d. **(c)** Also raise both sides of $d^{1/2} = t^{1/3}$ to the sixth power.

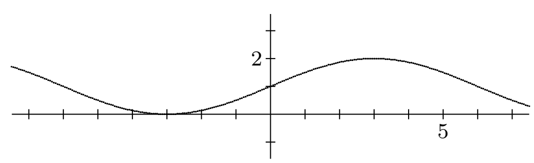
435. (Continuation) When distances are measured in astronomical units, Kepler's Third Law is $d^3 = t^2$. What would this equation look like if the distances were given in *miles*?

436. Convert the following to simpler equivalent forms:
(a) $x^6 x^{-6}$ **(b)** $(8a^{-3}b^6)^{1/3}$ **(c)** $\left(\dfrac{x^{1/2}}{y^{2/3}}\right)^6 \left(\dfrac{x^{1/2}}{y^{2/3}}\right)^{-6}$

437. Of all the points that lie on the plane $2x + 3y + 6z = 98$, which one is closest to the origin? Remember that the vector $[2, 3, 6]$ is perpendicular to the plane.

438. On one system of coordinate axes, graph the equations $y = 2^x$, $y = 3 \cdot 2^x$, $y = (0.4)2^x$, and $y = (-3)2^x$. What do all graphs $y = k \cdot 2^x$ have in common? How do they differ?

439. The sinusoidal graph at right touches the x-axis at -3. Write an equation that could have produced it. Does it affect your answer whether you choose to work in degree mode or radian mode?



440. In the xz-plane, $x^2 + z^2 = 9$ describes a circle. What does the same equation describe in xyz-space?

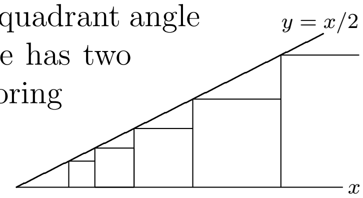
441. Consider the equations $2x + 3y + 5z = 15$ and $x - 2y + 2z = 3$. There are many points (x, y, z) whose coordinates fit *both* equations. Two of them are $(1, 1, 2)$ and $(17, 2, -5)$. Find another point whose coordinates fit both equations. What does the configuration of *all* the common solutions look like?

442. Let $F_1 = (-3, 0)$ and $F_2 = (3, 0)$, and let $P = (x, y)$ be a point on the graph of the equation $16x^2 + 25y^2 = 400$. Recall that PF_1 and PF_2 are the distances from P to F_1 and F_2, respectively. After you have verified that $PF_1 + PF_2 = 10$ holds whenever P is one of the four axis intercepts, calculate $PF_1 + PF_2$ for a fifth point P of your choosing. Solve $16x^2 + 25y^2 = 400$ for y, finding two functions that together generate this curve.

443. Let $F_1 = (-3, 0)$, $F_2 = (3, 0)$, and $P = (x, y)$. Use the distance formula to convert the equation $PF_1 + PF_2 = 10$ into *Cartesian form*. Then simplify the result by removing the radicals from the equation. To do this, start by arranging the terms so that there is one radical on each side of the equal sign. Then square both sides, as in $(PF_1)^2 = (10 - PF_2)^2$. Remember to make use of the identity $(m - n)^2 = m^2 - 2mn + n^2$. If all goes well, you will eventually discover that the equation $PF_1 + PF_2 = 10$ is equivalent to $16x^2 + 25y^2 = 400$.

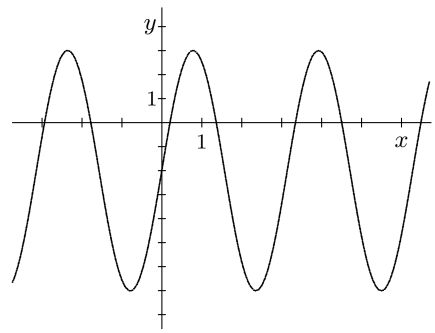
444. Explain why calculating $z^{2.5}$ is a square-root problem. What does $z^{0.3}$ mean?

445. The figure shows a sequence of squares inscribed in the first-quadrant angle formed by the line $y = \frac{1}{2}x$ and the positive x-axis. Each square has two vertices on the x-axis and one on the line $y = \frac{1}{2}x$, and neighboring squares share a vertex. The first (smallest) square is 8 cm tall. How tall are the next four squares in the sequence? How tall is the n^{th} square in the sequence?



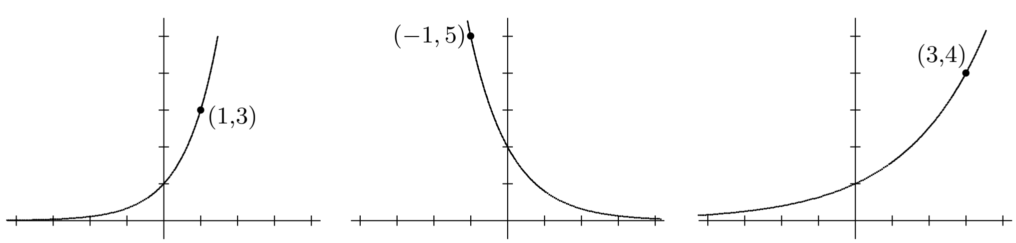
446. Verify that $(-8)^{1/3}$ can be evaluated, but that $(-8)^{1/4}$ cannot, and explain why $(-8)^{2/6}$ is ambiguous. To avoid difficulties like these, it is customary to restrict the base, b, of an exponential function, $f(x) = b^x$, to be a positive number.

447. The equation whose graph is shown at right has the form $y = k + a \sin 2x$. To three decimal places, one of the x-intercepts is 1.365. Working in radian mode, find three-place values for the other x-intercepts shown in the figure.



448. Suppose that a basket of groceries that costs \$68.80 today cost only \$53.20 a year ago. What is the annual rate of inflation? What is the *monthly* rate of inflation, assuming that it is constant?

449. Find a plausible equation $y = a \cdot b^x$ for each of the exponential graphs shown below:



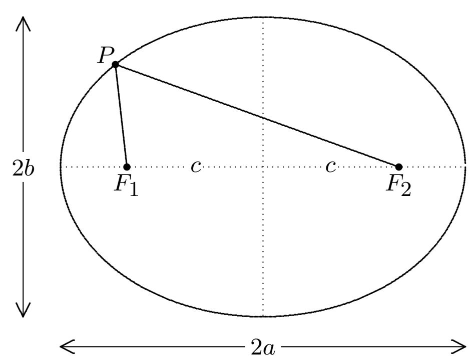
450. The value of $[\cos\theta, \sin\theta] \bullet [\cos(\beta+\theta), \sin(\beta+\theta)]$ does depend on the value of β, but does not depend on the value of θ. Explain why.

451. Graph the curve that is described parametrically by $(x, y) = (5\cos t, 4\sin t)$. If you are working in degree mode, the *parameter* interval should of course be $0 \le t \le 360$. This presentation should remind you of the parametric description of a circle. The curve is actually an ellipse. Confirm this by substituting the parametric equations into the ellipse equation $16x^2 + 25y^2 = 400$.

452. Write an equation for the plane that is perpendicular to the vector $[4, 7, -4]$ and that goes through the point $(2, 3, 5)$.

453. Write an equation of the form $y = k + a\sin mx$ for a curve that has a maximum point at $(30, 6)$ and that has 2 as its y-intercept. Use a graphing tool to check that the graph of your equation fits the given description. Because you have the freedom to work in either degree mode or radian mode, and because there are many curves that fit the given description, there are many correct answers to this question. Find another one.

454. *Some ellipse terminology.* For reasons that will eventually become clear, the anchor points F_1 and F_2 for the string are called *focal points* (or *foci*). The focal points are located on the *major* symmetry axis, with the ellipse *center* midway between them. The *vertices* are the points where the ellipse intersects the major axis. As shown, it is customary to let $2a$ be the distance between the vertices, $2c$ be the distance between the foci, and $2b$ be the distance between the intersections of the ellipse with the *minor* symmetry axis. As you know, $PF_1 + PF_2$ is constant for any point P on the ellipse. Explain why this constant equals $2a$. (*Hint*: Try a special position for P.) Each of PF_1 and PF_2 is called a *focal radius*.



455. (Continuation) Given $a = 12$ and $c = 8$, find b. (*Hint*: Try a special position for P.)

456. (Continuation) For any ellipse, what can be said about the quantity $b^2 + c^2$?

457. Make up a context for the equation $y = 5000(1.005)^x$.

458. (Continuation) Find the value of x that makes $y = 12500$. Find the value of x that makes $y = 2000$. Interpret these answers in the context you chose.

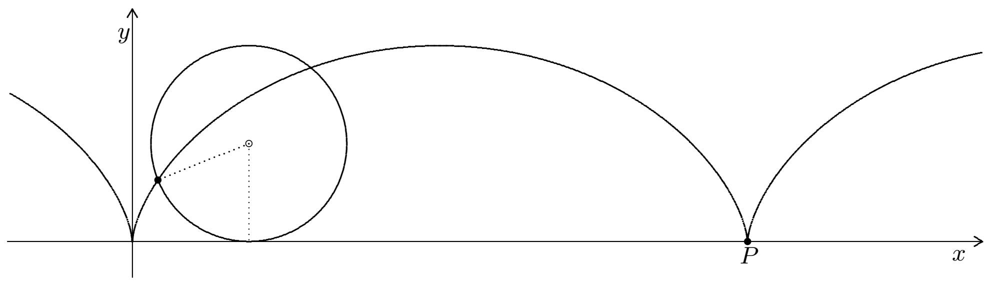
459. Write an equation for a plane that is perpendicular to the plane $2x - y + 3z = 6$ and that passes through the origin.

460. A constant monthly interest rate of 1.4% is equivalent to what annual interest rate?

461. Let $F_1 = (0, 2)$, $F_2 = (0, -2)$, and $P = (x, y)$. Use the distance formula to convert the equation $PF_1 + PF_2 = 6$ into Cartesian form. Simplify your answer until it reaches the form $hx^2 + ky^2 = m$.

462. (Continuation) Graph the curve described parametrically by $(x, y) = \left(\sqrt{5}\cos t, 3\sin t\right)$. Find a way of showing that this is the same ellipse that appeared in the preceding example.

463. *The cycloid.* A wheel of radius 1 rolls along the x-axis without slipping. A mark on the rim follows a path that starts at $(0, 0)$, as shown in the figure below.
(a) Find the x-coordinate of the point P where the mark first returns to the x-axis.
(b) Find both coordinates of the center after the wheel makes a quarter-turn.
(c) Find both coordinates of the mark after the wheel makes a quarter-turn.
(d) Find both coordinates of the mark after the wheel rolls a distance t, where $t < \frac{1}{2}\pi$.
(e) Check your formulas to see whether they are also correct for $\frac{1}{2}\pi \le t$.



464. Explain the notation $\mathbf{M}^{-1}$ used for the inverse of a square matrix $\mathbf{M}$.

465. Robin has some money invested in an account that pays 6 percent interest per year. At what rate is the investment increasing each *decade*? What is the *monthly* rate of growth?

466. The equation $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ describes an ellipse. Find the x- and y-intercepts of this curve, and use them to make a quick sketch. Once you have the values of a and b, use them to find c and add the focal points to your sketch. Finally, find a parametric description of this ellipse.

467. Write each of the following numbers as a power of 10. You should not need a calculator.
(a) 1000 **(b)** 1000000 **(c)** 0.01 **(d)** $\sqrt{10}$ **(e)** $100\sqrt{10}$ **(f)** $\dfrac{1}{\sqrt[3]{100}}$

468. If 2016 is written as a power of 10, its exponent is irrational. Use your calculator to find an approximation of this exponent. Then use that approximation (but *no* calculator) to write $\dfrac{1}{2016}$ as a power of 10.

469. (Continuation) Use a calculator to find $\log 2016$. Compare the displayed value with the first of your two previous answers.

470. Use a calculator to find the log of each of the following. Interpret the results. By the way, "log" is short for *logarithm*, to be discussed soon.
(a) 1000 **(b)** 1000000 **(c)** 0.01 **(d)** $\sqrt{10}$ **(e)** $100\sqrt{10}$ **(f)** $\dfrac{1}{\sqrt[3]{100}}$

471. Solve for x: $4^{2016} - 4^{2015} - 4^{2014} + 4^{2013} = 90\,(2^x)$

472. Explain your opinions of each of the following student responses:
(a) Asked for an expression equivalent to $x^3 + x^{-3}$, a student responded x^0.
(b) Asked for an expression equivalent to $\left(x^{-1} + y^{-1}\right)^{-2}$, a student responded $x^2 + y^2$.

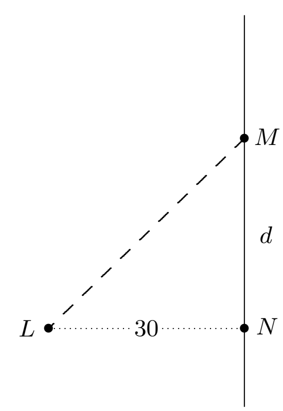
473. Using the log function, solve each of the following for x:
(a) $10^x = 3$ **(b)** $10^x = 300$ **(c)** $10^x = 9$ **(d)** $10^x = 3^{-1}$ **(e)** $10^x = \sqrt{3}$
You should see a few patterns in your answers — try to explain them.

474. The vertices of an ellipse are $(9, 0)$ and $(-9, 0)$, and the y-intercepts of the ellipse are 5 and -5. Write an equation for the ellipse.

475. Randy chooses a word in a dictionary and Andy tries to guess what it is, by asking questions to which Randy can answer only *yes* or *no*. There are 65 000 words in the dictionary. Show that Andy can guess the word by asking at most 16 questions.

476. Find the x- and y-intercepts of the generic ellipse described by $\dfrac{x^2}{m^2} + \dfrac{y^2}{n^2} = 1$. Find a parametric description of this ellipse.

477. A prison guard tower is 30 feet from the nearest wall of the prison. The diagram shows this arrangement from above, as if the viewer were in a helicopter. The spotlight L on top of the tower rotates counterclockwise, once every six seconds, casting a moving beam of light onto the wall. Let N be the point on the wall that is nearest the spotlight. Let M be the moving spot. Let d be the distance from N to M, and let t be the time, in seconds, since M last passed N. Find d when $t = 0.00$, $t = 0.30$, $t = 0.75$, and $t = 1.49$. Are d and t related linearly? What does the graph of this relationship look like? Why is "distance" not an appropriate word to use when describing the situation at the instant $t = 5$?



478. Rewrite
(a) the *logarithmic equation* $4 = \log 10000$ as an exponential equation;
(b) the exponential equation $10^{3.30449\cdots} = 2016$ as a logarithmic equation.

479. The function p defined by $p(t) = 3960(1.02)^t$ describes the population of Dilcue, North Dakota t years after it was founded.
(a) Find the founding population.
(b) At what annual rate has the population of Dilcue been growing?
(c) Calculate $p(65)/p(64)$.

480. (Continuation) Solve the equation $p(t) = 77218$. What is the meaning of your answer? By the way, notice that this question asks you to *find an exponent*. This is a typical logarithm question, for which you will soon learn a special technique.

481. Show that there is a dilation that transforms $9x^2 + 4y^2 = 36$ onto $9x^2 + 4y^2 = 225$. These ellipses are therefore *similar*. For each one, calculate c, a, and the ratio c/a, which is called the *eccentricity*. Notice that these ellipses have the same eccentricity. Explain why similar ellipses must have the same eccentricity. Is the converse true?

482. (Continuation) What are the possible values for the eccentricity of an ellipse? What is the eccentricity of a circle? What happens as the ellipse becomes less like a circle?

483. An ellipse has two symmetry axes. Why is one called *major* and the other called *minor*? Is it always true that the major axis is the x-axis?

484. The length of a piece of string is 18, and its ends are anchored at $(2, 6)$ and $(8, -2)$. An ellipse is traced with a pencil, while keeping the string taut. Sketch a graph of this curve, and find coordinates of its two vertices.

485. (Continuation) After learning about transforming points in a plane using rotation matrices and translation vectors, a curious Math 3 student wondered if the same could be done to find parametric equations for this ellipse.
(a) Determine the acute angle, θ, between the major axis of the ellipse and the x-axis.
(b) Write parametric equations for a congruent ellipse, centered at the origin, whose major axis is the x-axis.
(c) Orient the congruent ellipse appropriately by first rotating it by θ and then translating it. Do you want to use a matrix to rotate clockwise or counterclockwise?
(d) Use a graphing tool to sketch the ellipse.

486. (Continuation)
(a) If $P = (x, y)$, $F_1 = (2, 6)$, and $F_2 = (8, -2)$, write (though do not attempt to simplify) an equation that says $PF_1 + PF_2 = 18$. Sketch the graph using a graphing tool.
(b) As an algebra challenge, simplify the equation by hand so that it is in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where the coefficients are *relatively prime*.

487. Many ellipses are similar to $9x^2 + 25y^2 = 225$. Write an equation for the one whose focal points are:
(a) $(8, 0)$ and $(-8, 0)$; **(b)** $(0, 12)$ and $(0, -12)$.

488. Solve each of the following equations by hand.
(a) $8^x = 32$ **(b)** $27^x = 243$ **(c)** $1000^x = 100000$
Explain why all three equations have the same solution.

489. Given a positive number p, the solution to $10^x = p$ is called the *base-10 logarithm of* p, expressed as $x = \log_{10} p$, or simply $x = \log p$. For example, $10^4 = 10000$ means that 4 is the base-10 logarithm of 10000, or $4 = \log 10000$. The log function provides immediate access to such numerical information. Using a calculator for confirmation, and remembering that *logarithms are exponents*, explain why it is predictable that
(a) $\log 64$ is three times $\log 4$;
(b) $\log 12$ is the sum of $\log 3$ and $\log 4$;
(c) $\log 0.02$ and $\log 50$ differ only in sign.

490. In summarizing the growth of a certain population, Bailey writes $G(t) = 747t^{1.44}$ by mistake, instead of $747(1.44)^t$. Are there t-values for which the expressions agree in value?

491. Find an equation for the plane that goes through the point $(3, 2, 1)$ and that is parallel to the plane $7x + 4y + 4z = 65$.

492. An *exponential function* f is defined by $f(x) = k \cdot b^x$. Find k and b, given that
(a) $f(0) = 3$ and $f(1) = 12$ **(b)** $f(0) = 4$ and $f(2) = 1$

493. You have seen that an ellipse is in some ways like a circle. The following exercise, which illustrates an alternative definition, shows that an ellipse is in some ways also like a parabola: Let $F = (0, 0)$ be the *focus* and the line $y = -6$ be the *directrix.* Plot several points P that are *half as far from the focus as they are from the directrix.* The configuration of all such points is an ellipse. Identify the four points where the ellipse crosses the coordinate axes (two on each axis). Use the distance formula to write an equation for the ellipse.

494. (Continuation) Notice that this ellipse does not have the origin as its center, unlike the previous examples you have seen, so the x-axis is not a line of symmetry. The value of a for this ellipse should be clear, and the value of b can be obtained by finding where the ellipse meets its minor axis. Now calculate the value of c and notice that, had this ellipse been drawn by the string method, one end of the string would be held at the origin F.

495. (Continuation) Graph the curve that is described parametrically by the equation $(x, y) = (\sqrt{12}\cos t, 2 + 4\sin t)$, for $0 \le t \le 360$.

496. *What if the base of an exponential equation isn't 10*? One way of solving an equation like $1.02^x = 3$ is to use the log function to rewrite the equation as $(10^{0.0086})^x = 10^{0.4771}$. First justify this conversion, then solve $10^{0.0086x} = 10^{0.4771}$.

497. (Continuation) You have now calculated *the logarithm of 3 using the base 1.02*, for which $\log_{1.02} 3$ is the usual notation. The usual ways of reading $\log_{1.02} 3$ are "log base 1.02 of 3" or "log 3, base 1.02", or "the base-1.02 logarithm of 3", or "log to the base 1.02 of 3." The desired value was obtained as a quotient of two base-10 logarithms. Explain. By the way, do you recall a context for the equation $1.02^x = 3$?

498. On 24 August 1997 (which is the 236^{th} day of the year), the Fidelity Select Electronics Fund reported a 44.3 percent return on investments for the year to date. Calculate the annual growth rate for this fund in 1997. Explain your method and the assumptions you made.

499. Let $F = (9, 0)$ and choose a point P that fits the equation $16x^2 + 25y^2 = 3600$. Confirm that the distance from P to F is exactly three fifths the distance from P to the vertical line $x = 25$. Repeat this verification for two more points P that fit the equation. Calculate a and c for this ellipse, to show that its eccentricity is $3/5$. Hmm . . .

500. (Continuation) The line $x = 25$ is a directrix for the ellipse $16x^2 + 25y^2 = 3600$, and F is its focus. This ellipse has another focus G and another directrix. Find coordinates for G and an equation for its directrix. For any point P on the ellipse, what is the sum $PF + PG$?

501. Find the distance from the point $(1, 3, 5)$ to the plane $2x - y + 5z = 9$.

502. How are the graphs of $y = \cos x$ and $y = \cos(x - 90)$ related? Work in degree mode.

503. Imagine covering an unlimited plane surface with a single layer of pennies, arranged so that each penny touches six others tangentially. What percentage of the plane is covered?

504. For the first 31 days of your new job, your boss offers you two salary options. The first option pays you \$1000 on the first day, \$2000 on the second day, \$3000 on the third day, and so on — in other words, \$1000n on the n^{th} day. The second option pays you one penny on the first day, two pennies on the second day, four pennies on the third day — the amount doubling from one day to the next. Which option do you prefer, and why?

505. (Continuation) You have chosen the second payment option, and on the thirty-first day your boss pays you the wages for that day — in pennies. You wonder whether all these coins are going to fit into your dormitory room, which measures 12 feet by 15 feet by 8 feet. Verify that a penny is 0.75 inch in diameter, and that seventeen of them make a stack that is one inch tall. Use this information to decide whether the pennies will all fit.

506. Solve $2^x = 1000$. In other words, find $\log_2 1000$, the base-2 logarithm of 1000.

507. You now know how to calculate logarithms by using 10 as a *common base*. Use this method to evaluate the following. Notice those for which a calculator is not necessary.
(a) $\log_8 5$ **(b)** $\log_5 8$ **(c)** $\log_5 \sqrt{5}$ **(d)** $\log_{1.005} 2.5$ **(e)** $\log_3(1/9)$

508. Let $F = (0,0)$ be the focal point and λ (*Greek* "lambda") be the directrix $x = 5$. Plot a point P so that the distance from P to F is *two thirds* the distance from P to λ. The configuration of all such points P forms an ellipse. Find an equation for this curve, and make an accurate sketch of it, labeling key points (the vertices and the other focus) with their coordinates. Notice that the value of the eccentricity c/a for this ellipse is $2/3$, which equals *the distance ratio used to draw the curve*. It always works out this way (which can be proved as a supplementary exercise), thus there are two ways to think about eccentricity.

509. An exponential function f is defined by $f(t) = k \cdot b^t$. Its graph $y = f(t)$ contains the points $(1, 6)$ and $(3, 24)$. Find the constants k and b.

510. You have come to associate a function such as $p(t) = 450(1.08)^t$ with the size of something that is growing (exponentially) at a fixed rate. Could such an interpretation be made for the function $d(t) = 450 \cdot 2^t$? Explain.

511. An object is moving clockwise along the elliptical path $25x^2 + 4y^2 = 100$, making a complete tour every 20 seconds. The object is at $(0, 5)$ when $t = 0$.
(a) Write a parametric description of this motion, consistent with the given details.
(b) Do your equations describe an object that is moving with a constant speed? Explain.

512. The table shown at right contains the results of an experiment in which a golf ball is dropped from the roof of the Library. For each of the displayed times, the corresponding height of the ball above the ground is given. Use this data to answer the following questions:
(a) What was the *average speed* of the ball during the first second?
(b) What was the average speed of the ball during the half-second interval from $t = 0.5$ to $t = 1.0$?
(c) What was the average speed of the ball during each of the short time intervals from $t = 0.9$ to $t = 1.0$ and from $t = 1.0$ to $t = 1.1$?
(d) Approximately how fast was the ball traveling when $t = 1.0$?
(e) Approximately how fast was the ball traveling when $t = 1.8$?
(f) Approximately how fast was the ball traveling when $t = 2.2$?

t sec	h ft
0.0	80.0
0.1	79.8
0.2	79.4
0.3	78.6
0.4	77.4
0.5	76.0
0.6	74.2
0.7	72.2
0.8	69.8
0.9	67.0
1.0	64.0
1.1	60.6
1.2	57.0
1.3	53.0
1.4	48.6
1.5	44.0
1.6	39.0
1.7	33.8
1.8	28.2
1.9	22.2
2.0	16.0
2.1	9.4
2.2	2.6

513. Given $10^{0.301} = 2$ and $10^{0.477} = 3$, solve without a calculator:
(a) $10^x = 6$; **(b)** $10^x = 8$; **(c)** $10^x = 2/3$; **(d)** $10^x = 1$.

514. Given that $0.301 = \log 2$ and that $0.477 = \log 3$, you should not need a calculator to evaluate
(a) $\log 6$; **(b)** $\log 8$; **(c)** $\log(2/3)$; **(d)** $\log 1$.

515. What is the relationship between the graph of $y = \sin x$ and the graph of $y = \sin(x + 90)$? Work in degree mode.

516. Given that $m = \log a$, $n = \log b$, and $k = \log(ab)$,
(a) express a, b, and ab as powers of 10;
(b) use your knowledge of exponents to discover a relationship among m, n, and k;
(c) conclude that $\log(ab) = \log a + \log b$.

517. (Continuation) Justify the rules: **(a)** $\log(a^r) = r \log a$ **(b)** $\log(a/b) = \log a - \log b$

518. The function F defined by $F(x) = 31416(1.24)^x$ describes the number of mold spores found growing on a pumpkin pie x days after the mold was discovered.
(a) How many spores were on the pie when the mold was first discovered?
(b) How many spores were on the pie two days before the mold was discovered?
(c) What is the daily rate of growth of this population?
(d) What is the hourly rate of growth?
(e) Let $G(x)$ be the spore count on the same pie, x hours after the mold was discovered. Write a description of the function G.

519. The planes $x + 3y + 2z = 3$ and $4x + y - 2z = -5$ have the point $(2, -3, 5)$ in common. Justify this statement. The challenge is now to find coordinates for another point that lies on both planes. *Hint*: Because there are infinitely many correct answers to this question, *you will have to make an arbitrary choice during the solution process.* For instance, you could start by assigning one of the unknowns a value, and then use the given equations to solve for the other two unknowns.

520. Of all the points on a parabola, which one is closest to the focus? How do you know?

521. Write an equation of the form $y = k + a \cos mx$ for a curve that has a maximum point at $(20, 4)$ and that has -2 as its y-intercept. Use a graphing tool to check that your equation fits the given description. When you check answers with your neighbor, is it expected that you will both have found the same equation?

522. Another approach to solving an equation like $5^x = 20$ is to *calculate base-10 logarithms of both sides of the equation.* Justify the equation $x \log 5 = \log 20$, then obtain the desired answer in the form $x = \dfrac{\log 20}{\log 5}$. Evaluate this expression. Notice that $\log_5 20 = \dfrac{\log 20}{\log 5}$.

523. Write an expression for $\log_a N$ that refers only to base-10 logarithms, and explain.

524. Asked to simplify $\dfrac{\log 20}{\log 5}$, Brett replied "$\log 4$." What do you think of this answer?

525. The intersection of the planes $3x + 2y + 5z = 22$ and $2x + y + 3z = 13$ is a line, one of whose points is $(1, 2, 3)$. Find another point on this line. Once you have found a second point, describe *all* the points on the line by means of a parametric equation.

526. Explain why compounding a monthly inflation rate of 1% is *not* equivalent to an annual inflation rate of 12%.

527. A spherical balloon is being inflated by a machine. Starting from an initial volume of zero, the balloon's volume increases at a steady 1000 cc per second.
(a) Show that the balloon's radius after 6.0 sec of inflation is approximately 11.273 cm.
(b) What is the balloon's radius after 6.1 sec of inflation?
(c) Find approximately the rate (cm/sec) at which the radius is increasing at that instant when the volume reaches 6000 cc.

528. Given that $\log_c 8 = 2.27$ and $\log_c 5 = 1.76$, a calculator is *not needed* to evaluate
(a) $\log_c 40$ **(b)** $\log_c(5/8)$ **(c)** $\log_c 2$ **(d)** $\log_c(5^m)$ **(e)** $\log_c 0.04$

529. Consider the configuration of all points (x, y, z) for which x, y, and z are nonnegative and $x + y + z = 1$. This configuration is a familiar geometric object — what is it?

530. Consider the ellipse whose equation is $25x^2 + 9y^2 = 3600$.
(a) For this ellipse, find a, b, c, and the eccentricity.
(b) Find coordinates for the focal point F that is on the positive y-axis. Also find coordinates for a point P on the ellipse that is not an axis intercept.
(c) Find an equation for the directrix that corresponds to the focus F.

531. Sketch the plane whose equation is $3x + 5y + 7z = 15$. Include in your sketch the three lines along which the plane intersects the xy-plane, the yz-plane, and the xz-plane. Notice that the equation $3x + 5y + 7z = 15$ can be rewritten as $z = \frac{15}{7} - \frac{3}{7}x - \frac{5}{7}y$. What are the meanings of the three fractions that appear in this equation?

532. Given the equation $y = 1000x^2$, fill in the missing entries in the table at right, and enter this data into a calculator. Plot the data and notice that it is *not* linear. Create two new columns of transformed data, putting $\log x$ in one column and $\log y$ in the other. Plot the new columns against each other, and verify that they *are* related linearly. Could you have predicted the slope?

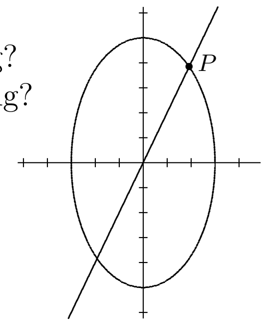
x	y
0.1	
1.0	
2.0	
3.0	
4.0	

533. Given $y = 1000x^2$, justify the equation $\log y = \log 1000 + \log(x^2)$. Then justify the equation $\log y = 3 + 2\log x$. Notice: y is not linearly related to x, but $\log y$ *is* linearly related to $\log x$.

534. Our civilization lies at the bottom of an ocean of air. As we move through this ocean, pressure varies exponentially with altitude — each increase of one mile of altitude causes the pressure to drop by 20 percent. Given that air pressure at sea level is 14.7 pounds per square inch, write formulas for the following exponential functions:
(a) $M(x)$ is the air pressure at an altitude of x miles above sea level.
(b) $F(x)$ is the air pressure at an altitude of x feet above sea level.
Then calculate the air pressure
(c) on the top of Mt Everest, which is 29141 feet above sea level;
(d) at the edge of the Dead Sea, which is 1300 feet *below* sea level.

535. Find a formula for a function f whose graph $y = f(x)$ has period 12 and y-values that vary between the extremes 2 and 8. How many such examples are there?

536. A spherical balloon is being inflated by a machine that increases the balloon's volume at a steady 1000 cc per second.
(a) Show that it takes 2.145 seconds for the balloon's radius r to reach 8.0 cm.
(b) How many seconds does it take for r to reach 8.1 cm?
(c) When r is 8.0 cm, at what approximate rate (cm/sec) is r increasing?
(d) When r is 16.0 cm, at what approximate rate (cm/sec) is r increasing?



537. Parametrized by $(x, y) = (3\cos t, 5\sin t)$, an object travels an elliptical path, crossing the line $y = 2x$ repeatedly. Describe all such t-values. For the intersection point P in the first quadrant, find the polar coordinates r and θ. Notice that θ is not equal to t.

538. Find a nonzero vector that is perpendicular to both of the vectors $[2, 3, 2]$ and $[4, 9, 5]$. Because there are infinitely many correct answers to this question, you will have to make an arbitrary choice somewhere during the solution process.

539. All of Casey's money is tied up in two funds. On 1 April, each fund was worth \$10000, but the value of the first fund was gaining 0.8% per month while the value of the second fund was losing 0.8% per month. Using this data, forecast the total value of Casey's holdings on 1 April of the next year.

540. *Logarithms would have changed Kepler's life (1571-1630).* The table at right contains astronomical data for the eight planets of our solar system. Each entry in the d-column stands for the mean distance from the planet to the sun, measured using the earth-to-sun mean distance (93 million miles) as the astronomical unit. Each entry in the t-column is the amount of time needed for the planet to make one revolution of its orbit, measured using earth years. Create two new columns of transformed data, putting the logarithms of the d-entries in one column and the logarithms of the t-entries in the other. Use the linear relationship between these variables to write a simple equation that expresses Kepler's Third Law. Notice how the use of logarithms reveals the relationship between d and t, and how it allows you to deal comfortably with numbers of disparate sizes (such as 0.241 and 164.8).

d	t
0.387	0.241
0.723	0.615
1.000	1.000
1.524	1.881
5.203	11.86
9.539	29.46
19.19	84.07
30.06	164.8

541. Discover a linear relationship between $\log x$ and $\log y$ by applying logarithms to both sides of the following:
(a) $y = x^3$ **(b)** $y = 100\sqrt{x}$ **(c)** $xy = 3960$ **(d)** $y^{2/3} = 5x^{5/2}$

542. Use exponential notation to rewrite **(a)** $x^2\sqrt{x}$ **(b)** $\dfrac{x}{\sqrt{x}}$ **(c)** $\dfrac{\sqrt{x}}{\sqrt[3]{x}}$ **(d)** $\sqrt[3]{x^3y^4z^5}$

543. A triangle has side lengths of 10, x, and $x + 4$. What are the possible values for x?

544. Ryan spills some soda and neglects to clean it up. When leaving for spring break, Ryan notices some ants on the sticky mess but ignores them. Upon returning seventeen days later, Ryan counts 3960 ants in the same place. The next day there are 5280 ants. Assuming that the size of the ant population can be described by a function of the form $F(t) = a \cdot b^t$, calculate the number of ants that Ryan saw when leaving for spring break.

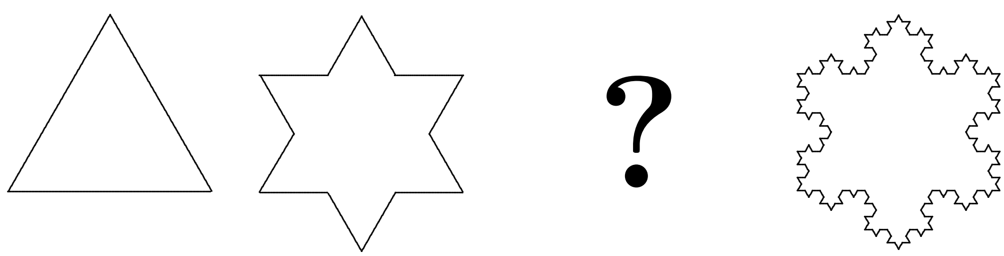
545. Given the equation $\log y = \log 3 + 4 \log x$, justify first that $\log y = \log 3 + \log(x^4)$, then that $\log y = \log(3x^4)$. Finally, write an equation that relates x and y and that makes no reference to logarithms. Notice that this equation expresses a *nonlinear* relationship between y and x, whereas the original equation expresses a *linear* relationship between $\log y$ and $\log x$.

546. Simplify the following equations by eliminating all references to logarithms

(a) $0.5 \log y + \log x = \log 300$ **(b)** $1.5 \log y = 2.699 - \log x$

547. A *geometric sequence* is a list in which each term is obtained by multiplying its predecessor by a constant. For example, 81, 54, 36, 24, 16, ... is geometric, with constant multiplier 2/3. The first term of this sequence is 81; what is the 40^{th} term? the millionth term? the n^{th} term? Check your formula for $n = 1$, $n = 2$, and $n = 3$.

548. In 1904 Helge von Koch invented his *snowflake*, which is probably the first published example of a *fractal*. It is the result of an endless sequence of stages: Stage 0 (the initial configuration) consists of an equilateral triangle, whose sides are 1 unit long. Stage 1 is obtained from stage 0 by replacing the *middle third* of each edge by a pair of segments, arranged so that a small equilateral triangle protrudes from that edge. In general, each stage is a polygon that is obtained by applying the middle-third construction to *every* edge of the preceding stage.



Stage	# of edges	# of vertices	edge length	perimeter	area
0	3	3	1	3	$\sqrt{3}/4$
1					

(a) Stages 0, 1, and 3 are shown above. Make your own sketch of stage 2.
(b) Stage 0 has three edges, and stage 1 has twelve. How many edges do stages 2 and 3 have? How many edges does stage n have? Continue the table begun above.
(c) Stage 1 has twelve vertices. How many vertices does stage n have?
(d) How long is each edge of stage 1? of stage 2? of stage n?
(e) What is the perimeter of stage 1? of stage 2? of stage n?
(f) Does the snowflake have finite perimeter? Explain.
(g) Is the area enclosed by the snowflake finite? Explain.

549. A triangle has side lengths $x+7$, x^2, and 13.
(a) If 13 is the largest side what are the possible values for x?
(b) If x^2 is the largest side what are the possible values for x?

550. Write an equation for the plane that contains the point $A = (4, 5, -3)$ and that is perpendicular to the line through $B = (5, -2, -2)$ and $C = (7, 1, 4)$.

551. The planes $-x + 2y + 3z = 4$ and $2x + 3y - z = 4$ intersect to form a line. Find an equation for this line.

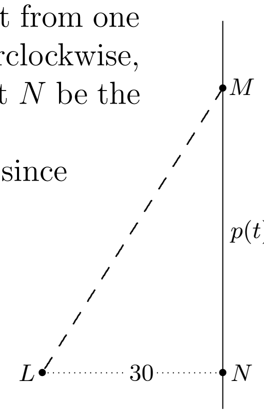
552. (Continuation) To the nearest tenth of a degree, find the size of the acute dihedral angle formed by the intersecting planes. *Hint*: You know an easy method of finding the angle formed by two vectors.

553. Without a calculator, find x: **(a)** $\log_4 x = -1.5$ **(b)** $\log_x 8 = 16$ **(c)** $27 = 8(x-2)^3$

554. Find the focal points and the eccentricity of the ellipse $x^2 + 4y^2 = 16$.

555. A triangle with a 32-inch side, a 40-inch side, and a 50-inch side is a curiosity, for its sides form a geometric sequence. Find the constant multiplier for this sequence. Find other such triangles. Are there any restrictions on the multipliers that can be used?

556. The top view at right represents a prison guard tower that is 30 feet from one of the prison walls. The spotlight L on top of the tower turns counterclockwise, once every six seconds, casting a moving beam of light onto the wall. Let N be the point on the wall nearest the spotlight. Let M be the moving spot.
(a) The position p of M relative to N is a function of t, the time elapsed since the spot was at N. Write a formula for $p(t)$ and discuss its domain.
(b) Calculate $p(0.5)$.
(c) What is the average speed of M during the first half-second?
(d) A hundredth of a second after M passes N, how far from N is M?
(e) Find the average speed of M during this hundredth of a second.



557. (Continuation) Calculate $p(0.24)$, $p(0.25)$, $p(0.2501)$, $p(0.251)$, and $p(0.26)$.
(a) Use $p(0.24)$ and $p(0.26)$ to estimate the speed of the spot at the instant $t = 0.25$.
(b) Use $p(0.25)$ and $p(0.251)$ to estimate the speed of the spot at the instant $t = 0.25$.
(c) Use $p(0.25)$ and $p(0.2501)$ to estimate the speed of the spot at the instant $t = 0.25$.
(d) Can you tell which estimate is closest to the actual speed? Or farthest from it?

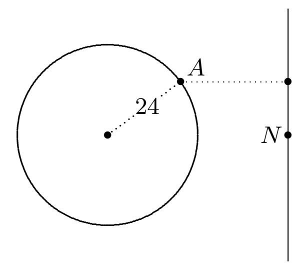
558. When $10^{3.43429448}$ is evaluated, how many digits are found to the left of the decimal point? You can answer this question without using a calculator, but you *will* need it to find the first three digits. What are the first three digits when $10^{9.43429448}$ is evaluated?

559. Asked to find an equation for the plane that contains points $A = (2, 3, 1)$, $B = (4, 1, 5)$, and $C = (3, 2, 4)$, Eugene answered, "$x + 3y + z = 12$." What do you think of Eugene's answer, and why?

560. Write an equation for the plane that is parallel to the plane $4x - 27y + 4z = 11$ and that contains the point $(6, 4, 95)$.

561. The fifth and sixth terms of a geometric sequence are 2880 and 1920, respectively. Find the seventh and first terms of this sequence.

562. Illuminated by the rays of the setting sun, Andy rides alone on a merry-go-round, casting a moving shadow on a wall. The merry-go-round is turning 40 degrees per second. As the top view shows, Andy is 24 feet from its center, and the sun's rays are perpendicular to the wall. Let N be the point on the wall that is closest to the merry-go-round. What is the speed (feet per second) of Andy's shadow when it passes N? What is the speed of this shadow when it is 12 feet from N?



563. (Continuation) Working in degree mode, graph the equation $y = 24 \sin 40x$. The point $P = (0.75, 12.00)$ is on this graph. What situation does P represent in the merry-go-round story? Find coordinates for another point Q on this graph that is very close to P, and find the slope of the line that goes through P and Q. What is the meaning of this slope in the story?

564. Find the base-10 logarithm of the large number 2^{216091}. To write out 2^{216091} in full, how many digits are needed? What are the first three (most significant) digits? By the way, $2^{216091} - 1$ was at one time the largest number known to be prime.

565. The planet Mercury follows an elliptical orbit of eccentricity 0.20, which — according to *Kepler's First Law* — has the sun at one focus. At their closest approach, Mercury and the sun are 48 million km apart. Make an accurate drawing of this orbit. What is the greatest distance between the sun and Mercury?

566. The table contains data collected by measuring the oscillation of home-made pendulums. Each entry in the L-column is the length of a pendulum (in centimeters) and each entry in the P-column is the period of oscillation (in seconds). Apply logarithms to both columns of data to help you find an equation that relates L and P. There is a long pendulum at the Museum of Science and Industry in Chicago, IL. How long is that pendulum, given that its period is measured to be 8.95 seconds?

L	P
15.0	0.78
18.0	0.85
22.0	0.94
33.0	1.15
40.0	1.27
45.0	1.34
55.0	1.48
70.0	1.67
90.0	1.90

567. Graph the curve $y = \frac{1}{4}(x + 2)(x - 6)$, and mark the point $F = (2, -3)$. Choose any point on the curve, and show that it is equidistant from F and the line $y = -5$. Do the same for three more points on the curve. These results suggest that the curve is the parabola whose *focus* is F and whose *directrix* is the line $y = -5$. What additional work is needed to prove this statement?

568. Consider the famous *Fibonacci sequence* 1, 1, 2, 3, 5, 8, 13, 21, … . Describe the pattern, calculate the next three terms, and explain why the sequence is not geometric.

569. Without a calculator, find: **(a)** $\log_5 125$ **(b)** $\log_{0.125} 64$ **(c)** $\log_4 8$ **(d)** $\dfrac{\log 125}{\log 5}$

570. What is half of 2^{40}? What is one third of 3^{18}?

571. Let $R(t) = 55(1.02)^t$ describe the size of the rabbit population in the PEA woods (known as the Gillespie Tract) t days after the first of June. Use a graphing tool to make a graph of this function inside the window $-50 \le t \le 100$, $0 \le R(t) \le 500$. What is the R-intercept of the graph, and what does it signify? Does the graph show a t-intercept? Would it show a t-intercept if the window were enlarged?

572. (Continuation) Choose a point on the graph that is very close to the R-intercept, then use these two points to estimate the rate (in rabbits per day) at which the population is growing on 1 June. In the same way, estimate the rate at which the population is growing on 1 September. Explain how your two answers relate to the given 2-percent growth rate.

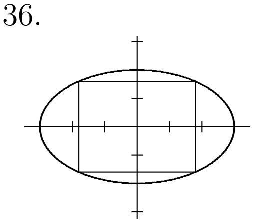
573. How deep is the water in a conical paper cup that is filled to half of its capacity?

574. Verify that the line $x + 2y = 8$ meets the ellipse $3x^2 + 4y^2 = 48$ at exactly one point, namely $P = (2, 3)$. The line is said to be *tangent* to the ellipse. Verify also that the focal points for this ellipse are $F_1 = (-2, 0)$ and $F_2 = (2, 0)$.

575. (Continuation) *The reflection property of the ellipse*: Use trigonometry to calculate the size of the acute angle formed by the tangent line and the focal radius F_2P. Do the same for the acute angle formed by the tangent line and the focal radius F_1P. How do your answers to these two calculations explain the title of this problem?

576. Write an equation to graph the top half of the ellipse $4x^2 + 9y^2 = 36$.

577. The figure at right shows one of the many rectangles that can be inscribed in the ellipse $4x^2+9y^2 = 36$. The first-quadrant corner of this rectangle is $(1.8, 1.6)$. Find the dimensions of the inscribed rectangle that has the largest *area*. It is not the one shown.



578. Find an equation for the plane tangent to the sphere $x^2 + y^2 + z^2 = 81$ at $(1, 4, 8)$.

579. The expansion of $\left(\dfrac{2}{3}\right)^{30}$ begins with $0.000\ldots$. How many zeros are there between the decimal point and the first nonzero digit? In the expansion of $\left(\dfrac{2}{3}\right)^{1\,000\,000}$ how many zeros are there between the decimal point and the first nonzero digit?

580. For the list of *triangular numbers* 1, 3, 6, 10, 15, 21, ..., describe the pattern, calculate the next three terms, and explain the terminology. Is this a geometric sequence? Explain.

581. Compare the graph of $y = \log x$ and the graph of $y = \log(10x)$. How are they related?

582. For any object that orbits the sun, Kepler's Third Law relates the *period* — the time needed for one orbit — and the *mean distance* from the sun — the average of the least and greatest distances (recall that the sun is at a focus). Halley's comet has a period of 76 Earth years (it next returns in 2061), and the least distance from the comet to the sun is 0.59 *astronomical unit* (one a.u. is about 93 million miles). Use Kepler's Third Law to calculate the mean distance and the greatest distance from the comet to the sun, and the eccentricity of the comet's orbit.

583. Estimate the slope of the curve $y = 2^x$ where it crosses the y-axis. The *slope of a curve* at a point P means the slope of the tangent line at P, and this line can be approximated by a *secant line* that goes through P and a nearby point on the curve.

584. Fill in the missing entries in the two tables shown at right. Do this without a calculator.

x	10^x
−3	
	1
0.5	3.162
	100
3	
	1996

x	$\log x$
0.001	
	0
3.162	
	2
1000	
1996	3.300

585. (Continuation) What do your results tell you about the graphs of $y = 10^x$ and $y = \log x$? It is customary to call functions that are related in this way *inverse functions*.

586. Without using a calculator, simplify **(a)** $\log(10^{-2.48})$ **(b)** $10^{\log 4.8}$

587. Justify the equivalence of $(x, y) = (2 + 5\cos t, -1 + 3\sin t)$ and $\dfrac{(x-2)^2}{25} + \dfrac{(y+1)^2}{9} = 1$, by first solving the parametric equations for $\cos t$ and $\sin t$, then squaring both sides.

588. Suppose that the logarithms of two quantities H and k are related by the equation $\log H = 1.48 - 2.5\log k$. Relate these two quantities by an equation that makes no reference to logarithms. Sketch a graph of $\log H$ versus $\log k$. On a separate set of axes sketch a graph of H versus k. Recall that a graph of a versus b means a is plotted on the vertical axis and b is plotted on the horizontal axis.

589. If there exist constants a, b and c such that $a \cdot p + b \cdot q = c$, then the variable quantities p and q are *linearly related.* For example, the equation $2.5\log k + \log H = 1.48$ shows that $\log k$ and $\log H$ are linearly related with $a = 2.5$, $b = 1$ and $c = 1.48$.

(a) Are x and $\log y$ linearly related if $x = 2.5 + \dfrac{\log y}{3}$?

(b) Are x and y linearly related if $6xy = 10$?

590. Suppose that $\log y$ and $\log x$ are linearly related such that $a\log y + b\log x = c$. Show that y can be expressed as a *power function* of x. In terms of a, b, and c, find this function.

591. Ming is wondering how to distinguish a power function from an exponential function. How would you explain the difference?

592. Show that the graph of the quadratic equation $y = x^2$ is a parabola, by finding coordinates for its focus and an equation for its directrix.

593. The focal points of an ellipse are $(12, 0)$ and $(-12, 0)$, and the point $(12, 7)$ is on the ellipse. Find the points where this curve intersects the coordinate axes.

594. Forty-eight dice are rolled and each die that shows a "2" (a deuce) on top is removed. The remaining dice are rolled again and the deuces are removed. This procedure is repeated until all the dice are gone. How many rolls will be needed? Explain.

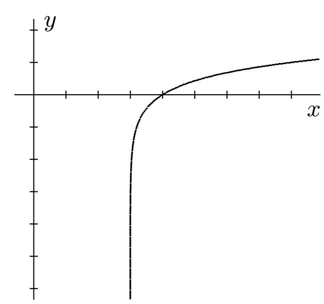
595. Draw the graph of the equation $y = \log_2 x$. How does this graph compare to the graph of the equation $y = 2^x$?

596. Give equations for two circles that are *concentric*. Give equations for two ellipses that are *confocal* (which means that their focal points are shared). Are confocal ellipses necessarily concentric? Why?

597. How does the graph of $(x-3)^2+(y+1)^2 = 121$ compare with the graph of $x^2+y^2 = 121$?

598. How does the graph of $\dfrac{(x-3)^2}{16} + \dfrac{(y+1)^2}{9} = 1$ compare with the graph of $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$?

599. The equation graphed at right is $y = \log_5(x-3)$. What is the x-intercept of this graph? There are many vertical lines that do not intersect this graph; which one of them is farthest to the right? For what x-values does the equation make sense? What x-value corresponds to $y = 1$? to $y = 2$? to $y = 3$?



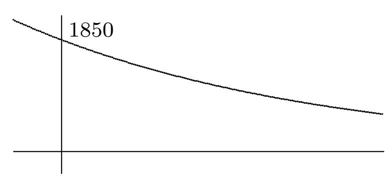
600. (Continuation) How does the given graph compare to the graph of $y = \log_5 x$? How does the given graph compare to the graph of $y = \log_5(x+2)$?

601. Find equations for the directrices of the ellipse $3x^2 + 4y^2 = 12$.

602. Let $f(x) = 1850(0.96)^x$. Find a context to interpret your answers to each of the next two questions:

(a) Calculate $\dfrac{f(2.01) - f(2.00)}{0.01}$.

(b) Solve the equation $f(x+1) = f(x) - 25$.



603. The first two terms of a geometric sequence are 1850 and 1776. What is the forty-third?

604. Write an equation for the curve obtained by shifting the curve $y = 2^x$ **(a)** 3 units to the right; **(b)** 5 units down; **(c)** 3 units to the right *and* 5 units down. Identify x- and y-intercepts and other significant features.

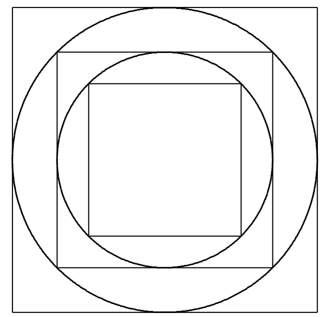
605. Find the y-intercept of the graph of $y + 1 = 2^{x-3}$. How does the graph of $y + 1 = 2^{x-3}$ compare with the graph of $y = 2^x$? How about the graph of $y = 2^{x-3} - 1$?

606. How does the graph of $y = f(x)$ compare to the graph of $y + 1 = f(x-3)$?

607. Verify that the line $8x + y = 98$ is tangent to the ellipse $4x^2 + 3y^2 = 588$ at $P = (12, 2)$. Verify also that the focal points for this ellipse are $F_1 = (0, -7)$ and $F_2 = (0, 7)$. Calculate the size of the acute angle formed by the tangent line and the focal radius F_2P. Do the same for the acute angle formed by the tangent line and the focal radius F_1P.

608. A big pile of gravel contains 10^{10} stones. This pile is separated into two smaller piles by repeating the following process until the original pile is gone: Take ten stones from the original pile, then throw nine of them onto the left pile and one onto the right pile. When the original pile is gone, how many stones will be in the left pile? the right pile?

609. The figure at right shows an outermost 1×1 square, within which appears an inscribed circle, within which appears an inscribed square, within which appears another inscribed circle, within which appears another inscribed square. Although the figure does not show it, this process can be continued indefinitely. Let $L_1 = 1$ be the length of a side of the first (largest) square, L_2 be the length of a side of the second square, L_3 be the length of a side of the third square, and so on. Show that the numbers L_1, L_2, L_3, ... form a geometric sequence, and calculate L_{20}.



610. (Continuation) Let A_n be the area of the n^{th} square. How is A_2 related to A_1? How is A_3 related to A_2? How is A_{n+1} related to A_n? This last equation, together with the equation $A(1) = 1$, is said to be a *recursive description* of the sequence A_n.

611. Find the focus and the directrix for the parabola $y = ax^2$. What special meaning does the expression $\dfrac{1}{4a}$ have?

612. What is the x-intercept of the graph of $y = \log(x - 3) - 1$? How does the graph of $y = \log(x - 3) - 1$ compare with the graph of $y = \log x$?

613. Write 10^{1997} as a power of 2.

614. Simplify by hand:

(a) $(3^{-1} + 4^{-1})^{-1}$ **(b)** $\dfrac{6^{4000}}{12^{2000}}$ **(c)** $7^u 7^u$ **(d)** $\sqrt{64x^{16}}$ **(e)** $2^m 3^{-m}$

615. A six-sided die is to be rolled once. What is the probability of obtaining a "2" (a *deuce*)? What is the probability of obtaining a non-deuce? The same die is to be rolled three times. What is the probability of obtaining three deuces? three non-deuces?

616. There are many examples of graphs that have period 60 and whose y-values vary between -8 and 8. First, find a formula for such a function f, given that $f(0) = 8$. Next, find a formula for such a function g, given that $g(0) = 0$.

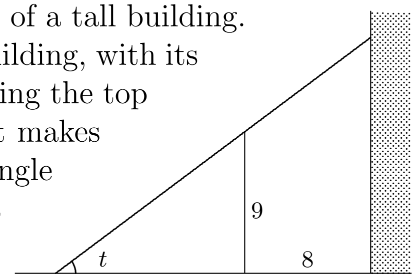
617. (Continuation) For your f, find the three smallest positive x that make $f(x) = 5$.

618. Show that the sphere $(x-5)^2+(y+2)^2+(z-4)^2=36$ and the plane $2x+y+2z=34$ have exactly one point in common. *Hint*: vectors, hmm. . . .

619. A function of the form $H(x)=a\cdot b^x$ has the property that $H(1)=112$ and $H(3)=63$. Find the values $H(0)$ and $H(4)$.

620. Given that $\log_4 x$ is somewhere between -1.0 and 0.5, what can be said about x?

621. A fence that is 9 feet tall is situated 8 feet from the side of a tall building. As the figure at right shows, a ladder is leaning against the building, with its base outside the fence. It so happens that the ladder is touching the top of the fence. Find the length of the ladder, given that **(a)** it makes a 60-degree angle with the ground; **(b)** it makes a t-degree angle with the ground. **(c)** Graph the function you found in part (b) and find the length of the shortest ladder that reaches the building from outside the fence.



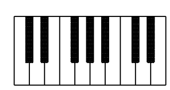
622. Show that the following equations are parabolas by finding foci and directrices for each.
(a) $y=2x^2$ **(b)** $y=2x^2+5$ **(c)** $y=2x^2+12x+5$

623. What is the distance between the parallel planes
(a) $y=-1$ and $y=5$?
(b) $4x+3y=7$ and $4x+3y=27$?
(c) $4x+7y+4z=33$ and $4x+7y+4z=87$?

624. Find a focus and a directrix for the ellipse $4x^2+3y^2=108$.

625. The numbers 3, 12, 48, 192, ... form a geometric sequence. What can be said about the sequence of *logarithms* of these numbers? Does it have any special property? Does it make any difference what base is used for the logarithms?

626. When the note *middle C* is struck on a piano, it makes a string vibrate at 262 cycles per second. When the corresponding note one octave higher (denoted C′) is struck, it makes a string vibrate at 524 cycles per second (twice as fast as the first string). These two numbers form part of a geometric sequence of frequencies

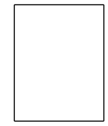


$$\ldots, \text{C}, \text{C}\#, \text{D}, \text{D}\#, \text{E}, \text{F}, \text{F}\#, \text{G}, \text{G}\#, \text{A}, \text{A}\#, \text{B}, \text{C}', \ldots$$

known as *equal-tempered tuning.* Given C = 262 and C′ = 524, find the frequency of G, and the frequency of the note that is n steps above middle C. (For example, F is 5 steps above C. By the way, a musician would say "half-step" instead of "step.")

627. (Continuation) Without calculating any of the indicated frequencies, explain why the ratio G:C is the same as the ratio A:D, which is the same as the ratio B:E. Now calculate the ratio G:C and show that it is approximately equal to 3:2, the musical interval known as a *fifth.* The disagreement between the two ratios is why some violinists do not like to make music with pianists.

628. A rectangular sheet of paper (such as the one shown in the figure at right) has thickness 0.003 inches. Suppose that it is folded in half, then folded in half again, then folded in half again — fifty times in all. How thick is the resulting wad of paper?



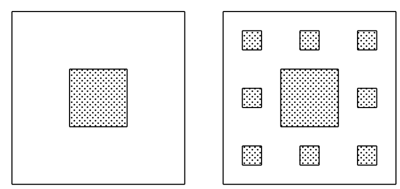
629. (Continuation) Give a recursive description for the thickness of the paper.

630. The point $P = (-3, 2.4)$ is on the ellipse $9x^2 + 25y^2 = 225$. Verify this, then come as close as you can to finding the slope of the line that is tangent to the ellipse at P.

631. (Continuation) Points on the ellipse can be described parametrically by the equation $(x, y) = (5\cos t, 3\sin t)$. Find a t-value that produces P. Then find a t-value that produces a point on the ellipse that is very close to P.

632. The point $(3, 8)$ is on the graph $y = 2^x$. What is the corresponding point on the graph of the inverse function? Find four more pairs of points like these.

633. A blank, square sheet of paper is painted as follows: Step 1 consists of ruling the square into nine smaller congruent squares, and painting the central one. Step 2 consists of applying this process to each of the remaining eight squares (a small central square is painted in each). Step 3 consists of applying this process to each of the 64 remaining small squares, etc. The first two steps are shown.



Let A_0 be the area of the original square, A_1 be the area that is unpainted after Step 1, A_2 be the area that is unpainted after Step 2, and so on. Show that the decreasing sequence A_0, A_1, A_2, … is geometric. Is it ever true that $A_n < 0.01A_0$? Justify your answer.

634. Consider the function $f(x) = x^3$.
(a) Write a formula for $g(x)$, the inverse of $f(x)$.
(b) Choose three points on the graph of $y = f(x)$, at least one of them with a negative x-coordinate. What are their corresponding points on the graph of $y = g(x)$?
(c) What geometrical transformation can be applied to the graph of $y = f(x)$ in order to generate the graph of the inverse function?

635. Morgan is going to roll a six-sided die five times. What is the probability that only the final roll will be a deuce?

636. Verify that the line $y = 3x - 9$ is tangent to the parabola $4y = x^2$ at $P = (6, 9)$. Verify also that the focal point for this parabola is $F = (0, 1)$. Calculate the size of the acute angle formed by the tangent line and the segment FP. Do the same for the acute angle formed by the tangent line and the line $x = 6$.

637. The equations $U_{1950} = 25\,791\,017$ and $U_n = U_{n-1} + 0.027U_{n-1}$ provide a recursive description of a population that has an annual growth rate of 2.7%, where U_n is the size of the population at the beginning of year n.
(a) Explain why the numbers U_n form a geometric sequence.
(b) Write explicit equations for U_{1951}, U_{1952} , U_{2000}, and U_{1950+k}. An explicit description for f_n is different from a recursive description because it does not require the evaluation of another f_k. For example, $f_n = 1.05f_{n-1}$, with $f_0 = 10$ is recursive while $f_n = 10(1.05)^n$ is explicit.
(c) Find the first integer n for which $100\,000\,000 \le U_n$.

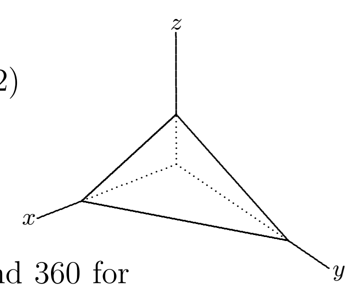
638. There are twenty-three red marbles and two blue marbles in a box. A marble is randomly chosen from the box, its color noted, then put back in the box. This process is repeated. What is the probability that **(a)** the first marble is blue? **(b)** the first four marbles are red? **(c)** the first four marbles are red and the fifth is blue?

639. Let $f(x) = 5^x$. Calculate $\dfrac{f(1.003) - f(1.000)}{0.003}$ and then explain its significance.

640. How does the graph of $y = \left(\frac{1}{2}\right)^x$ compare with the graph of $y = 2^x$? What features do these curves have in common? How are the slopes of these curves related at their common y-intercept?

641. Let $R(t) = 55(1.02)^t$ describe the size of the rabbit population in the PEA woods t days after the first of June. Let $B(t) = 89(1.01)^t$ describe the size of the beaver population in the same area t days after the first of June.
(a) When are there the same number of rabbits as beavers? At what rate (rabbits per day) is the rabbit population increasing then? At what rate is the beaver population increasing?
(b) Solve the equation $R(t) = 2B(t)$. Interpret your answer.

642. The plane that contains the points $(8, 3, 1)$, $(2, 6, 3)$, and $(4, 6, 2)$ has an equation of the form $ax + by + cz = d$. Find coefficients for this equation, trying *two* different approaches to the problem. One method uses vectors, another does not.



643. Working in degree mode, find both values of n between 0 and 360 for which $\sin 7672 = \sin n$. A calculator is not needed.

644. On the same system of coordinate axes, graph the circle $x^2 + y^2 = 25$ and the ellipse $9x^2 + 25y^2 = 225$. Draw the vertical line $x = 2$, which intersects the circle at two points, called A and B, and which intersects the ellipse at two points, called C and D. Show that the ratio $AB:CD$ of chord lengths is 5:3. Choose a different vertical line and repeat the calculation of the ratio of chord lengths. Finally, using the line $x = k$ (with $|k| < 5$, of course), find expressions for the chord lengths and show that their ratio is 5:3. Where in the diagram does the ratio 5:3 appear most conspicuously? Because the area enclosed by the circle is known to be 25π, you can now deduce the area enclosed by the ellipse.

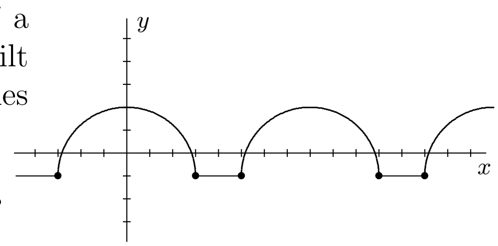
645. (Continuation) What is the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$?

646. For any positive base b, the graph $y = b^x$ intersects the y-axis at $(0, 1)$. The slope m of the curve at this intersection depends on b, however. For example, you have probably already found that m is about 0.693 when $b = 2$. What is the (approximate) value of m when $b = 3$?

647. (Continuation) Make a table that includes (at least) the b-entries 1, 2, 3, 4, 6, 1/2, 1/3, and 1/4, and their corresponding m-entries. By the way, it is possible to save some work by writing your m-approximation formula in terms of b.

648. (Continuation) There are some familiar patterns in the table. Have you ever seen another table of values that exhibits this pattern? Make a scatter plot of the data. Can this nonlinear relationship be straightened?

649. The figure at right shows the graph $y = f(x)$ of a periodic function. The graph, whose period is 8, is built from segments and semicircular arcs. Notice the values $f(3) = -1$ and $f(5) = -1$.
(a) Calculate $f(320)$, $f(323)$, and $f(558)$.
(b) What does the graph of $y + 1 = f(x - 3)$ look like?



650. Show that it is possible to use logarithms to solve the equation $x^{2.5} = 1997$. Then show that it is not necessary to do so.

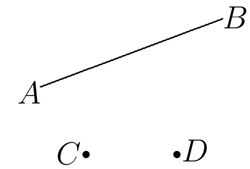
651. Just as it governs the satellites of the sun, Kepler's Third Law governs the satellites of Earth; the largest one is the moon. The center of the moon is about 239000 miles from the center of the earth, and it takes 27.3 days to complete one orbit. How far from the center of the earth must a satellite be, if it takes exactly *one day* to complete one orbit?

652. (Continuation) Show that placing three satellites in one-day orbits (which are called *geosynchronous orbits*) enables communication between almost any two points on Earth.

653. A balloon is losing its helium at 12 percent per day. Given that the balloon initially held 5000 cc of helium, write a complete recursive description of the balloon's volume.

654. Verify that $A = (-3, 4)$ is on the circle $x^2 + y^2 = 25$ and $P = (-3, 2.4)$ is on the ellipse $9x^2 + 25y^2 = 225$. Find an equation for the line tangent to the circle at A. Then show how this can be used to find an equation for the line tangent to the ellipse at P.

655. It can be shown that there is a unique point P on line AB that makes the sum $CP + PD$ as small as possible and simultaneously makes the angles APC and BPD the same size. To show this reflect C across AB to C'. Draw $C'D$ and label P the intersection of $C'D$ and AB. Let Q be any point on AB other than P.



(a) Show that $CQ + QD > CP + PD$.
(b) Show that angles AQC and BQD are not congruent.

656. *Reflection property of the ellipse*. Suppose that P is a point on an ellipse whose focal points are F_1 and F_2. Draw the intersecting lines PF_1 and PF_2, as well as the bisectors of the four angles they form. Consider the bisector that does not separate F_1 and F_2. Prove that given any point Q other than P on this line, $QF_1 + QF_2 > PF_1 + PF_2$. Explain why the line meets the ellipse *only* at P. Justify the title of this problem.

657. Two intersecting lines form four angles. The bisectors of these angles have a special property. Show how this property can be helpful when finding lines tangent to ellipses.

658. Jamie is riding a Ferris wheel that takes fifteen seconds for each complete revolution. The diameter of the wheel is 10 meters and its center is 6 meters above the ground.
(a) When Jamie is 9 meters above the ground and rising, at what rate (in meters per second) is Jamie gaining altitude? **(b)** When is Jamie rising most rapidly? At what rate?

659. Solve for y: **(a)** $(1.5)^y = 3.6(1.25)^y$ **(b)** $\begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19 \\ 96 \end{bmatrix}$ **(c)** $6x^2 + 9y^2 = 144$

660. The moons of Jupiter are still being discovered (28 known as of 2000). Galileo found the first four of them in 1610. Europa is 671000 km from the center of Jupiter and completes one orbit every 3.551 days. What is the distance from the center of Jupiter to Io, whose period is 1.769 days? Ganymede and Callisto are 1070000 km and 1880000 km from the center of Jupiter, respectively. What are their periods?

661. A simple example of a *rate* is the speed of a car, for example 55 *miles per hour*. Bankers also speak of interest *rates*, as in a savings account whose annual interest rate is 4 *percent*. What are the actual *units* for this rate? You hardly ever hear or see them expressed.

662. The point $P = (6, 5)$ is on the ellipse $5x^2 + 9y^2 = 405$. Verify this and make a sketch. Using two different methods, find an equation for the line that intersects the ellipse tangentially at P.

663. The height of an object moving up and down is described by $y = 72 + 40\cos 18t$ (in degree mode). This is an example of *simple harmonic motion*. To at least three decimal places, confirm that the *average speed* of the object is $2/\pi$ times its *greatest speed*.

664. The Hoyts company has a fleet of 1000 cars for one-day local rentals, which must be leased from either the Exeter office or the Hampton office. The company expects that 70% of the cars rented in Exeter return to the Exeter office, 60% of the cars rented in Hampton return to the Hampton office, and all cars return to one office or the other. Assume that all 1000 cars are rented each day, and that 200 of the rental cars are in Exeter and 800 are in Hampton on Monday morning.
(a) What distribution of cars is expected on Tuesday morning?
(b) What distribution of cars is expected on Wednesday morning?

665. (Continuation) Verify that the equations

$$\begin{aligned} E_0 &= 200 \qquad & E_n &= 0.70E_{n-1} + 0.40H_{n-1} \\ H_0 &= 800 \qquad & H_n &= 0.30E_{n-1} + 0.60H_{n-1} \end{aligned}$$

provide a recursive description of the rental-car distribution.
(a) What information do E_n and H_n convey when $n = 1$?
(b) Verify that E_n and H_n could be written in matrix form as

$$\begin{bmatrix} E_n \\ H_n \end{bmatrix} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} E_{n-1} \\ H_{n-1} \end{bmatrix}$$

666. (Continuation) Let $\mathbf{v} = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$ be the initial distribution of cars and $\mathbf{M} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$. Thus $\mathbf{Mv}$ is the distribution of cars on Tuesday morning.
(a) What does the matrix product $(\mathbf{M})\mathbf{Mv} = \mathbf{M}^2\mathbf{v}$ represent?
(b) What does the matrix product $\mathbf{M}^{10}\mathbf{v}$ represent?
(c) Matrix $\mathbf{M}$ is an example of a *transition matrix*. After you label its rows and columns, interpret the entries of $\mathbf{M}$ as probabilities.
(d) The sum of the entries in each column of $\mathbf{M}$ is 1; how is this significant?

667. (Continuation) Solve the equation $\mathbf{M} \begin{bmatrix} x \\ 1000 - x \end{bmatrix} = \begin{bmatrix} x \\ 1000 - x \end{bmatrix}$. Interpret.

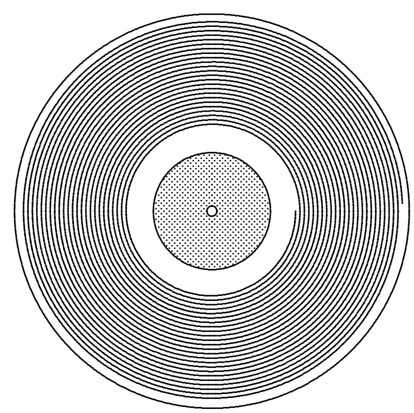
668. *The reflection property of parabolas.* Let F be the focus of a parabola, and let P be an arbitrary point on the parabola. Let μ be the line through P that is parallel to the axis of symmetry of the parabola; this means that μ intersects the directrix perpendicularly at a point N. Let λ be the perpendicular bisector of FN.
(a) Explain why P is on λ.
(b) Explain why λ is tangent to the parabola.
(c) Explain why λ bisects angle FPN.
(d) Explain why it makes sense for a car headlight or a spotlight to have a bulb at the focus of a parabolic reflector.
(e) Explain why it makes sense for a solar oven, a satellite dish, or a parabolic microphone to have a parabolic reflector.

669. What else can be said about the positive number p, given that
(a) $0.0 < \log_7 p < 1.0$? **(b)** $0.0 < \log_{1/2} p < 1.0$? **(c)** $0.0 < \log_b p < 1.0$?

670. An atom of carbon-14 is unstable, meaning that it can spontaneously transform itself (by radioactive decay) into nitrogen at any instant. The probability that this will actually happen to a specific atom of carbon-14 in the course of a year is only 0.0121 percent, however. In other words, there is a 99.9879 percent chance that any specific atom of carbon-14 will survive another year. Suppose that one million carbon-14 atoms are placed in a container. How many of these atoms are expected to be carbon-14 atoms one year later? The half-life question: How much time will it take for half the atoms to decay?

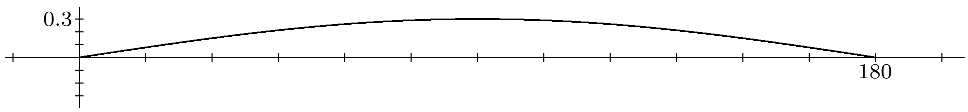
671. (Continuation) Carbon-14, which is produced from nitrogen when solar radiation bombards the upper atmosphere, makes up about 1 trillionth of all the carbon found in things that rely on air to live. When an organism dies, it no longer takes in air, and its supply of carbon-14 diminishes exponentially as described above. Apply this principle to estimate the age of a lump of charcoal (found in a cavern at an ancient campsite) that has only 32 percent as much carbon-14 as it had when the charcoal was still firewood.

672. A typical long-playing phonograph record (once known as an LP) plays for about 24 minutes at 33 1/3 revolutions per minute while a needle traces the long groove that spirals slowly in towards the center. The needle starts 5.7 inches from the center and finishes 2.5 inches from the center. Estimate the length of the groove.



673. Courtney is supposed to run laps around the outdoor track. At the start of each lap, however (even the first), there is always an 8 percent chance that Courtney will call it quits for the day. What is the probability that today Courtney will run **(a)** no laps? **(b)** at least four laps? **(c)** exactly four laps?

674. Interpreted in degree mode, the equation $y = 0.3\cos(36000t)\sin(x)$ models the motion of a stretched, vibrating string that is 180 centimeters long ($x = 0$ is one end of the string and $x = 180$ is the other). The amplitude of the vibration is 0.3 cm.
(a) Verify that the period of the vibration is 0.01 second. This means that the string vibrates 100 times per second.



(b) Taking a photo of the string corresponds to choosing a t-value. For example, the diagram shows the string at the instant $t = 0$. Notice that the amplitude has been exaggerated to make it more visible. Sketch photos of the string for $t = 0.001$, $t = 0.0025$, $t = 0.003$, and $t = 0.005$ second.
(c) How far does the center of the string move during one complete vibration? Does the center of the string move with constant speed? Calculate an approximate value for the speed of the center of the string when $t = 0.0025$ second.

675. All students at PEA go to *Las Olas* or *Lexie's* once a week. Each week, 66% of the students who went to *Lexie's* the previous week decide to go to *Las Olas* and 27% of the students who went to *Las Olas* the previous week decide to go to *Lexie's.* Assume that this trend continues indefinitely and that the student population of PEA numbers 1000. If 420 students went to *Las Olas* last week, how many students should *Lexie's* expect this week? In three weeks? In ten weeks? In the long run?

676. (Continuation) In setting up #675, Sam considered the following possible matrix equations, using X for *Lexie's* and O for *Las Olas.* Which of these represent the data presented in #675?

(**a**) $\begin{bmatrix} X_n \\ O_n \end{bmatrix} = \begin{bmatrix} 0.34 & 0.66 \\ 0.27 & 0.73 \end{bmatrix} \begin{bmatrix} X_{n-1} \\ O_{n-1} \end{bmatrix}$ (**b**) $\begin{bmatrix} X_n \\ O_n \end{bmatrix} = \begin{bmatrix} 0.34 & 0.27 \\ 0.66 & 0.73 \end{bmatrix} \begin{bmatrix} X_{n-1} \\ O_{n-1} \end{bmatrix}$

(**c**) $\begin{bmatrix} O_n \\ X_n \end{bmatrix} = \begin{bmatrix} 0.73 & 0.66 \\ 0.27 & 0.34 \end{bmatrix} \begin{bmatrix} O_{n-1} \\ X_{n-1} \end{bmatrix}$ (**d**) $\begin{bmatrix} O_n \\ X_n \end{bmatrix} = \begin{bmatrix} 0.66 & 0.34 \\ 0.27 & 0.73 \end{bmatrix} \begin{bmatrix} O_{n-1} \\ X_{n-1} \end{bmatrix}$

677. (Continuation) Explain why Val's equation below is an accurate representation of the data in #675, yet not as useful as either of the correct equations in #676.

$$\begin{bmatrix} X_n \\ O_n \end{bmatrix} = \begin{bmatrix} 0.27 & 0.34 \\ 0.73 & 0.66 \end{bmatrix} \begin{bmatrix} O_{n-1} \\ X_{n-1} \end{bmatrix}$$

678. Earthquakes can be classified by the amount of energy they release. Because of the large numbers involved, this is usually done logarithmically. The Richter scale is defined by the equation $R = 0.67\log(E) - 1.17$, where R is the rating and E is the energy carried by the seismic wave, measured in kilowatt-hours. (A kilowatt-hour is the energy consumed by ten 100-watt light bulbs in an hour).
(a) The 1989 earthquake in San Francisco was rated at 7.1. What amount of energy did this earthquake release? It could have sustained how many 100-watt light bulbs for a year?
(b) An earthquake rated at 8.1 releases more energy than an earthquake rated at 7.1. How many times more?
(c) Rewrite the defining equation so that E is expressed as a function of R.
(d) Adding 1 to any rating corresponds to multiplying the energy by what constant?
(e) Is it possible for a seismic wave to have a *negative* rating? What would that signify?

679. On 1 July 2012, you deposit 1000 dollars into an account that pays 6 percent interest annually. How much is this investment worth on 1 July 2032?

680. (Continuation) On 1 July 2013, you deposit 1000 dollars into an account that pays 6 percent interest annually. How much is *this* investment worth on 1 July 2032? Answer the same question for deposits made on 1 July 2014, 1 July 2015, and so forth, until you see a pattern developing.

681. (Continuation) Suppose that you deposit 1000 dollars into the same account on 1 July *every* year. The problem is now to calculate the combined value of *all* these deposits on 1 July 2032, including the deposit made on that final day. Rather than getting the answer by tediously adding the results of *twenty-one* separate (but similar) calculations, we can find a shorter way. Let V stand for the number we seek, and observe that

$$V = 1000(1.06)^0 + 1000(1.06)^1 + \cdots + 1000(1.06)^{19} + 1000(1.06)^{20}$$

is the very calculation that we wish to avoid. Obtain a second equation by multiplying both sides of this equation by 1.06, then find a way of combining the two equations to obtain a compact, easy-to-calculate formula for V.

682. (Continuation) Any list *first*, *first* $\cdot$ *multiplier*, *first* $\cdot$ *multiplier*^{2}, ..., in which each term is obtained by multiplying its predecessor by a fixed number, is called a geometric *sequence*. A geometric *series*, on the other hand, is an *addition problem* formed by taking consecutive terms from some geometric sequence. Two examples: $16 + 24 + 36 + 54$ is a four-term geometric series whose sum is 130, and $32 - 16 + 8 - 4 + \cdots + 0.125$ is a nine-term geometric series whose sum is 21.375. Consider now the typical geometric series, which looks like $\textit{first} + \textit{first}\cdot\textit{multiplier} + \textit{first}\cdot\textit{multiplier}^2 + \cdots + \textit{last}$. Find a compact, easy-to-calculate formula for the sum of all these terms.

683. Write the sum of $3383(1.04)^0 + 3383(1.04)^1 + 3383(1.04)^2 + \cdots + 3383(1.04)^n$ compactly.

684. Convert the following equations into non-logarithmic forms:
(a) $2.3\log x + \log y = 1.845$ **(b)** $\log y = 3.204 - 0.510x$

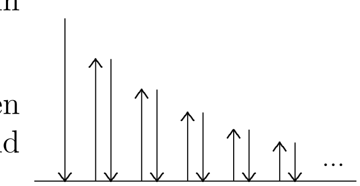
685. Find an equation for the plane that contains the points $P = (3, 1, 4)$, $Q = (1, 4, -2)$, and $R = (9, -1, 1)$. There is more than one possible form for your answer, and more than one workable approach. You will have to make some choices.

686. The point $P = (-12, 2)$ is on the ellipse $4x^2 + 3y^2 = 588$. Find an equation for the line through P that is tangent to the ellipse.

687. The lunchtime crowd in Exeter frequents either *The Green Bean* or *Me & Ollie's*. Competition between the two restaurants is high, and customer surveys disclose that 35% of those who eat at the *Me & Ollie's* will switch to *The Green Bean* the next day, while 28% of those who eat at *The Green Bean* will switch to *Me & Ollie's* the next day. Given that 50% of the lunchtime crowd ate at each of the two restaurants today, predict the distribution of customers **(a)** tomorrow **(b)** the day after tomorrow **(c)** three weeks from now. This sequence of lunchtime distributions is an example of a *Markov chain*.

688. Find equivalent expressions:
(a) $\dfrac{(2.86)^x}{(1.43)^x}$ **(b)** $(1.4)^x + (1.4)^{x+1} + (1.4)^{x+2}$ **(c)** $\dfrac{\log k}{\log m}$

689. The third term of a geometric sequence is 40. The sixth term is 135. What is the seventh term of this sequence?



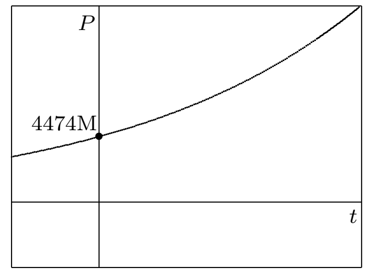
690. A speckled green superball has a 75% rebound ratio. When you drop it from a height of 16 feet, it bounces and bounces and bounces . . .
(a) How high does the ball bounce after it strikes the ground for the third time?
(b) How high does the ball bounce after it strikes the ground for the seventeenth time?
(c) When it strikes the ground for the second time, the ball has traveled a total of 28 feet in a *downward* direction. Verify this. How far downward has the ball traveled when it strikes the ground for the seventeenth time?

691. (Continuation) At the top of its second rebound, the ball has traveled 21 feet *upward*.
(a) At the top of its seventeenth rebound, how far upward has the ball traveled?
(b) At the top of its seventeenth rebound, how far has the ball traveled in total?

692. Consider the function $f(x) = x^2$. Does it have an inverse? If so, write a formula for it; if not, why not? In considering this question, please draw a graph of $y = f(x)$ as well as its reflection over the line $y = x$.

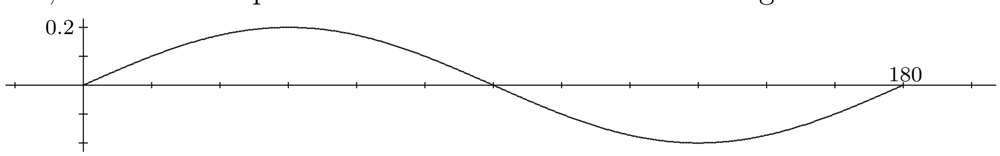
693. Sometimes it is necessary to invest a certain amount of money at a fixed interest rate for a fixed number of years so that a financial goal is met. The initial amount invested is called the *present value* and the goal is called the *future value*. The parents of a child born today decide that \$350 000 will be needed for college expenses. They find a certificate of deposit that pays 0.5 percent interest each month. How much (present value) should they invest so that there is \$350 000 on the child's 18^{th} birthday?

694. The size of the earth's human population at the beginning of the year $1980+t$ can be approximated by exponential functions. One such function is $P(t) = 4\,474\,469\,000(1.0140)^t$. Its graph is shown at right. At what rate is $P(t)$ increasing when **(a)** $t = 10$ and when **(b)** $t = 30$? There are at least two ways to interpret each of these questions. Provide an answer for each of your interpretations.



695. (Continuation) Does it make sense to say that the earth's human population is growing at a *constant rate*? Discuss.

696. Interpreted in degree mode, the equation $y = 0.2\cos(72000t)\sin(2x)$ models the motion of a stretched, vibrating string that is 180 centimeters long ($x = 0$ is one end of the string and $x = 180$ is the other). The amplitude of the vibration is 0.2 cm.
(a) Verify that the period of the vibration is 0.005 second. This means that the string vibrates 200 times per second.
(b) Taking a photo of the string corresponds to choosing a t-value. For example, the diagram below shows the string at the instant $t = 0$. Notice that the amplitude has been exaggerated to make it more visible. Sketch photos of the string for $t = 0.001$, $t = 0.0025$, $t = 0.003$, and $t = 0.005$ second.
(c) How far does the center of the string move during one complete vibration?
(d) What two points on the string move the most during one complete vibration? For either one of them, calculate its speed at the instant when it is crossing the x-axis.



697. Many times you have used the function $\sin^{-1}$ to find angles, trusting (for example) that calculators will always return an angle whose sine is 0.3 when you request $\sin^{-1}(0.3)$. Now it is time to consider the rule that makes $\sin^{-1}$ a function.
(a) The point $(30, 0.5)$ is on the graph of $y = \sin x$ (working in degree mode). There is a corresponding point that is on the graph of $y = \sin^{-1} x$. What is it?
(b) Write four other sets of corresponding points — two with negative x-coordinates and two with positive x-coordinates.
(c) Notice that the point $(150, 0.5)$ is on the graph of $y = \sin x$ but the point $(0.5, 150)$ is not on the graph of $y = \sin^{-1} x$. Explain.

698. Draw the graph of $x = \sin y$. Explain why it is not the same as the graph of $y = \sin^{-1} x$.

699. Explain why the expression $\log(a) - \log(b)$ should not be confused with $\log(a - b)$. Rewrite $\log(a) - \log(b)$ in an equivalent logarithmic form.

700. The human eardrum responds to a very wide range of loudness, tolerating intensities as great as $10^{12}I_0$, where I_0 is the intensity of a barely audible sound. The loudness of a sound whose intensity is I is said to be $10 \cdot \log(I \div I_0)$ *decibels.* On this scale, threshhold sounds $I = I_0$ have decibel level 0, while the loudest tolerable sounds are rated at 120. Intensity is often measured in watts per square meter.
(a) A loud stereo produces sound that is a hundred million times as intense as a threshhold sound. What is the decibel level of such sound?
(b) Suppose that two loud stereos are simultaneously playing the same music, side-by-side. What is the decibel level of their combined sound?
(c) The decibel level of a buzzing mosquito is 40. What is the decibel level of a gang of one thousand identical buzzing mosquitos?

701. Multiply each of the following by $1 - r$:
(a) $1 + r$ **(b)** $1 + r + r^2$ **(c)** $1 + r + r^2 + \cdots + r^{1995}$

702. Find sums for the following geometric series:
(a) $3000 + 3150 + \cdots + 3000(1.05)^{20}$ **(b)** $18 - 12 + 8 - \cdots + 18(-2/3)^{49}$

703. (Continuation) It is often convenient to use what is called *sigma notation* to describe a series. For example, the preceding parts (a) and (b) can be described by

$$\sum_{k=0}^{20} 3000(1.05)^k, \text{ and either } \sum_{n=1}^{50} 18\left(-\frac{2}{3}\right)^{n-1} \text{ or } \sum_{p=0}^{49} 18\left(-\frac{2}{3}\right)^p,$$

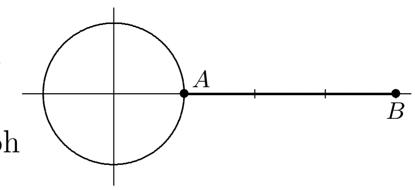
respectively. The symbol Σ is the Greek letter *sigma*, which corresponds to the English S (for *sum*). Evaluate the series (c) and (d), then express (e) and (f) using sigma notation:
(c) $\displaystyle\sum_{k=0}^{9} 12\left(\frac{3}{5}\right)^k$ **(d)** $\displaystyle\sum_{n=4}^{7} 5n$ **(e)** $1 + 4 + 9 + \cdots + 361$ **(f)** $8 + 4 + 2 + \cdots + \frac{1}{4} + \frac{1}{8}$
Notice that some of these series are *not* geometric. By the way, calculators have the built-in capacity to evaluate sigma notation.

704. The rebound ratio of a speckled green superball is 75%. It is dropped from a height of 16 feet. Consider the instant when ball strikes the ground for the fiftieth time.
(a) How far downward has the ball traveled at this instant?
(b) How far (upward and downward) has the ball traveled at this instant?
(c) How far would the ball travel if you just let it bounce and bounce and bounce ... ?

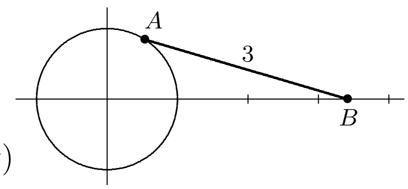
705. (Continuation) Explain why the following are equivalent.
(a) $\displaystyle\sum_{n=0}^{49} 16(0.75)^n$ **(b)** $\displaystyle\sum_{n=1}^{50} 16(0.75)^{n-1}$ **(c)** $\displaystyle 16 + \sum_{n=1}^{49} 12(0.75)^{n-1}$ **(d)** $\displaystyle 16 + \sum_{n=0}^{48} 12(0.75)^n$

706. In most mathematics books, the notation $\sin^2 A$ is often used in place of the clearer $(\sin A)^2$, or $\cos^3 B$ in place of $(\cos B)^3$. Why do you think that writers of mathematics fell into this strange habit? It is unfortunate that this notation is *inconsistent* with notation commonly used for inverse functions. Explain.

707. A wheel of radius 1 meter is centered at the origin, and a rod AB of length 3 meters is attached at A to the rim of the wheel. The wheel is turning counterclockwise, one rotation every 4 seconds, and, as it turns, the other end $B = (b, 0)$ of the rod is constrained to slide back and forth along a segment of the x-axis. Given any time t seconds, the position of B is determined. This functional relationship is expressed by writing $b = f(t)$. The top figure shows this apparatus when $t = 0$ and $b = 4$, and the bottom figure corresponds to $t = 0.64$.



(a) Calculate $f(0)$, $f(1)$, $f(2)$, and $f(11)$. Calculate $f(t)$ for a t-value that you and *only* you choose.
(b) Explain why f is a *periodic* function of t, and sketch a graph of $b = f(t)$ for $0 \le t \le 8$. What is the range of values of f?



708. (Continuation) Calculate a general formula for $f(t)$, and use it to refine your sketch.
(a) Solve the equation $f(t) = 3$.
(b) In degree mode, compare the graph of $g(t) = 3 + \cos(90t)$ with the graph of f.
(c) Do you think that the graph of the periodic function $f(t)$ is *sinusoidal*? Explain.
(d) Why was this particular $g(t)$ chosen?

709. (Continuation) What is the *velocity* of B when $t = 1$? Find an instant t of time when you think B is moving most rapidly. Explain your choice.

710. You have used the function $\tan^{-1}$ many times to find angles. Now it is time to consider the rule that makes $\tan^{-1}$ a function.
(a) The point $(45, 1)$ is on the graph of $y = \tan x$ (working in degree mode). There is a corresponding point that is on the graph of $y = \tan^{-1} x$. What is it?
(b) Write four other sets of corresponding points — two with negative x-coordinates and two with positive x-coordinates.
(c) Notice that the point $(135, -1)$ is on the graph of $y = \tan x$ but the point $(-1, 135)$ is not on the graph of $y = \tan^{-1} x$. Explain.

711. In order that the sequence $9\ ,\ x\ ,\ 16\ ,\ \ldots$ of positive numbers be geometric, what must x be? This value of x is called the *geometric mean of 9 and 16.*

712. Matrix multiplication is a complicated process. You might therefore be surprised by the results of the following two calculations:

(a) $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} \left(\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 0 & 3 \end{bmatrix} \right)$ **(b)** $\left(\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \right) \begin{bmatrix} -2 & 6 \\ 0 & 3 \end{bmatrix}$

Test this apparent coincidence on three random square matrices of your own choosing. Is this phenomenon familiar to you?

713. Often trigonometric functions can be simplified.
(a) Working in degree-mode, graph $y = \sin(x + 90)$. On the basis of your graph, suggest a simpler expression that is equivalent to $\sin(x + 90)$.
(b) Write the radian-mode version of (a) and answer it.

714. Assume that N.H. Lottery commissioners invest their revenue at 8% annual interest.
(a) To make a \$50 000 payment in one year, how much money should they invest now?
(b) To make a \$50 000 payment in four years, how much money should they invest now?
(c) To award a \$500 000 prize in ten \$50 000 annual payments, the first of which is due immediately, how much money should the Lottery commissioners invest now?

715. Pat and Kim are talking on the telephone during a thunderstorm. After one of the lightning flashes, Pat hears the rumble of thunder *twice* — the first sound coming through the open window, and the second sound coming *through the telephone* ten seconds later. Given that Pat lives two miles east of the center of town, Kim lives two miles west of the center of town, both on the same east-west road, and that sound takes five seconds to travel a mile through air, draw a map that shows some of the places where the lightning could have struck. For example, could the lightning have struck the road on which Pat and Kim live? Assume that light and electricity take no time to reach their destinations.

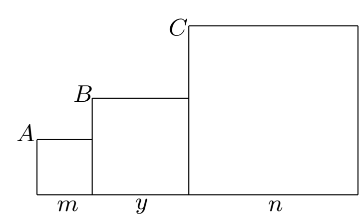
716. The *domain* of the function $\sin^{-1}$ consists of all numbers between -1 and 1, inclusive. What is the domain of the function $\cos^{-1}$? What is the domain of the square-root function? What is the domain of log? What is the domain of an exponential function?

717. When working in degree mode, the *range* of the function $\sin^{-1}$ consists of all numbers between -90 and 90, inclusive. What is the range of the function $\cos^{-1}$? What is the range of the square-root function? What is the range of log? What is the range of $f(x) = 2-7\sin 45x$? What is the range of the exponential function $E(x) = b^x$?

718. Simplify the following expressions: **(a)** $\log_c A + \log_c \dfrac{1}{A}$ **(b)** $p + pm + pm^2 + \cdots + pm^w$

719. Draw the graph of $x = \tan y$, then explain why it is not the same as the graph of $y = \tan^{-1} x$.

720. Three squares are placed next to each other as shown. The vertices A, B, and C are collinear. In terms of m and n, find the dimension y.



721. Show that $y = 3\cos 2x$ can be rewritten in the equivalent form $y = a\sin(mx + b)$, thereby confirming that this cosine curve is sinusoidal.

722. There are two boxes. The red box contains four red marbles and one blue marble, and the blue box contains five blue marbles and three red marbles. The following experiment is done a hundred times: A marble is randomly drawn from one of the boxes, its color recorded, then replaced in the same box. The first drawing is from the red box, but each subsequent drawing is determined by the color of the marble most recently drawn: if it is red, the next drawing comes from the red box; if it is blue, the next drawing comes from the blue box. What is the probability that the first marble is red? the first marble is blue? the second marble is red? the second marble is blue? the hundredth marble is red?

723. Pat lives two miles east of the center of town, Kim lives two miles west of the center of town, both on the same east-west road. They are talking on the telephone during a thunderstorm. Fifteen seconds after a lightning flash, Pat hears the rumble of thunder. Ten seconds after that, Pat hears the *same* peal of thunder through the telephone. Because sound takes five seconds to travel a mile through air, and light and electricity take no time to reach their destinations, Pat reasons that there are only two places where the lightning could have struck. What is the reasoning? Could Kim have reached the same conclusion?

724. (Continuation) Which equation, $LK - LP = 2$ or $LP - LK = 2$, best fits the situation that you have been investigating?

725. *Spreading rumors.* It is tempting to think that rumors spread like populations grow. In other words, the more who know a rumor, the more who will learn of it during the next hour. This does not take diminishing opportunity into account, however: The more who know a rumor, the *fewer* there are to tell it to. A simple quadratic model for this phenomenon is obtained by assuming that the number who learn of a rumor each hour is proportional to the number who already know it, and *also* proportional to the number who do not. For example, suppose that three PEA students know a certain rumor at 7 am, that there are 1000 students in all, and that any student will try to pass a rumor along to one more student each hour. This suggests the model $S_n = S_{n-1} + \dfrac{1000 - S_{n-1}}{1000}\, S_{n-1}$, where S_n is the number of students who know the rumor n hours after 7 am. It is given that $S_0 = 3$. The fraction in the recursive equation represents that portion of the student body that does not know the rumor. Calculate S_1, S_2, and S_3. Your answers will suggest the equation $S_n = S_{n-1} + S_{n-1}$, which describes 100% growth. Explain why this model becomes less realistic as n increases.

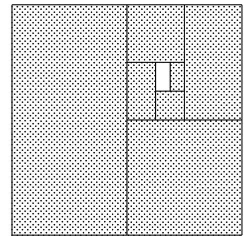
726. (Continuation) It is convenient to apply this equation in a different form, in which S_n is replaced by $1000P_n$. In other words, P_n is the fractional part $S_n/1000$ of the student body that knows the rumor after n hours. After these replacements are made, the recursion simplifies to $P_n = P_{n-1} + (1 - P_{n-1})\, P_{n-1}$, and the initial condition is $P_0 = 0.003$.

(a) Use the recursion and the initial condition to calculate what percentage of the student body knows the rumor by 5 pm. What percentage know by ten o'clock check-in?

(b) Plot the ordered pairs (n, P_n) for $n = 0, 1, 2, \ldots, 15$.

(c) Which hourly increase $P_n - P_{n-1}$ is the greatest?

(d) If each student told *two* students the rumor, how would this affect the recursion?



727. What does the figure at right suggest to you about the geometric series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$?

728. Let $F_1=(-2,0)$, $F_2=(2,0)$, and $P=(x,y)$. Write an equation that says that $PF_1-PF_2=2$ and make a sketch. What does this have to do with Pat and Kim? Starting with the step $(PF_1)^2=(2+PF_2)^2$, simplify your equation until it reaches the form $hx^2-ky^2=m$.

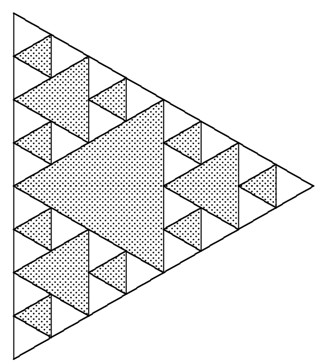
729. Consider the function f defined by $f(x)=|\sin x|$. Draw its graph, and find the period. Explain why this graph is not sinusoidal.

730. If $p(t)$ is an exponentially decreasing function, what is the usual name for the number t that solves the equation $p(t)=\frac{1}{2}p(0)$? What is the usual name for the number u that solves the equation $p(37+u)=\frac{1}{2}p(37)$?

731. A calculator has a sin function and a $\sin^{-1}$ function, a cos function and a $\cos^{-1}$ function, and a tan function and a $\tan^{-1}$ function. Does it have a $\log^{-1}$ function?

732. The repeating decimal $0.13131313\ldots$ can be thought of as an infinite geometric series. Write it in the form $a+ar+ar^2+\cdots$. Express the series using sigma notation, and then find its sum. What is the rational number equivalent to $0.13131313\ldots$?

733. Find the solution to $\log_3(T)=\frac{1}{2}$, then find all the solutions to $\log_3(\tan x)=\frac{1}{2}$.



734. An equilateral triangle of unit area is painted step-by-step as follows: Step 1 consists of painting the triangle formed by joining the midpoints of the sides. Step 2 then consists of applying the same midpoint-triangle process to each of the three small unpainted triangles. Step 3 then consists of applying the midpoint-triangle process to each of the nine very small unpainted triangles. The result is shown at right. In general, each step consists of applying the midpoint-triangle process to each of the (many) remaining unpainted triangles left by the preceding step. Let P_n be the area that was painted *during* step n, and let U_n be the total unpainted area left after n steps have been completed.

(a) Find U_1, U_2, U_3, P_1, P_2, and P_3.

(b) Write a recursive description of U_n in terms of U_{n-1}. Find an explicit formula for U_n.

(c) Write a recursive description of P_n in terms of P_{n-1}. Find an explicit formula for P_n.

(d) Use your work to evaluate the sum $\dfrac{1}{4}+\dfrac{3}{16}+\dfrac{9}{64}+\cdots+\dfrac{3^{99}}{4^{100}}+\dfrac{3^{100}}{4^{101}}$.

(e) Express the series of part (d) using sigma notation.

735. An NBA center recently signed a seven-year contract for \$121 million. What is the present value of this contract, given a 9% interest rate? In other words, how much must the ball club invest when the contract is signed, so that it can make seven equal payments of \$17285714.29, the first one due immediately?

736. Function f is defined by $f(x) = \sin^{-1}(\sin x)$. Draw its graph, and find its period.

737. Given that the period of function f is 6, and the period of function g is 4, what can be said about the periodicity of the function h defined by $h(x) = f(x) + g(x)$?

738. Graph $\dfrac{x^2}{1} - \dfrac{y^2}{3} = 1$ in windows of various sizes. Include the small window determined by $-5 \le x \le 5$, $-5 \le y \le 5$ and the large window with $-100 \le x \le 100$, $-100 \le y \le 100$. Describe the significant features of this graph, which is an example of a *hyperbola*.

739. The part of the hyperbola, $\dfrac{x^2}{1} - \dfrac{y^2}{3} = 1$, above the x-axis is described by $y = \sqrt{3x^2 - 3}$. Choose two first-quadrant points on the graph of $y = \sqrt{3x^2 - 3}$ that have large x-values, and calculate the slope of the line they determine. You should find that the slope is close to 1.732. Now notice that $\sqrt{3x^2 - 3}$ is virtually the same as $x\sqrt{3}$ when x is large and positive. Give an explanation. Because the line $y = x\sqrt{3}$ becomes indistinguishable from the hyperbola, it is called an *asymptote* for the hyperbola, and the two graphs are said to be *asymptotic*.

740. Write down the first few terms of any geometric sequence of positive terms. Make a new list by writing down the logarithms of these terms. This new list is an example of what is called an *arithmetic sequence*. What special property does it have?

741. *A new look at a familiar hyperbola*: Draw the vertical line $x = 1/2$, and let F be $(2, 0)$. Verify that the distance from the point $Q = (2, 3)$ to F is twice the distance from Q to the line $x = 1/2$. Find other points that have the same property. Write an equation that describes all such points $P = (x, y)$. By analogy with ellipses, this hyperbola is said to have *eccentricity* 2. Explain.

742. The data at right displays how the temperature (in degrees Celsius) of a cup of coffee diminishes as time (in minutes) increases. *Newton's Law of Cooling* says that the rate at which an object's temperature changes is proportional to the difference between the temperature of the object and the ambient (surrounding) temperature, which is 20 degrees in this example. Create a third column of data by subtracting 20 from each entry in the second column. Create a fourth column of data by dividing each entry in the third column by the entry immediately above it. What conclusion do you draw from the entries in the fourth column? Use your findings to write a formula that describes the temperature of the coffee as a function of time.

time	*temp*
0.0	90.0
5.0	76.2
10.0	64.6
15.0	55.8
20.0	48.3
25.0	42.8
30.0	38.7
35.0	34.7
40.0	31.9
45.0	29.1
50.0	27.5
55.0	26.3
60.0	24.5
65.0	23.7

743. Given that $f(x) = \sin(x)\cos(x)$, find the period of function f, and the range of its values. Do you think that f is sinusoidal? Explain.

744. Without using a calculator, simplify the following (in degrees):
(a) $\sin\left(\sin^{-1} 0.32\right)$ **(b)** $\tan\left(\tan^{-1} 1.61\right)$ **(c)** $\cos^{-1}(\cos 123)$ **(d)** $\sin^{-1}(\sin 137)$

745. By means of a formula, invent a periodic but *non-sinusoidal* function f, whose period is 10, and whose values $f(x)$ oscillate between the extremes 5 and 9.

746. Courtney is about to start running laps on the track. At the beginning of each lap, there is an 8% chance that Courtney will decide to stop for the day, and a 92% chance that another lap will be run. Thus the probability that Courtney runs *exactly* two laps is $(0.92)(0.92)(0.08)$. Give two justifications for the equation

$$1 = (0.08) + (0.92)(0.08) + (0.92)^2(0.08) + (0.92)^3(0.08) + \cdots$$

747. (Continuation) Given that p and q are positive numbers whose sum is 1, find the sum of the infinite series $q + pq + p^2q + p^3q \cdots$.

748. Let $F_1 = (0, -5)$ and $F_2 = (0, 5)$. The equation $|PF_1 - PF_2| = 8$ is satisfied by many points $P = (x, y)$. Identify and plot the two points P that are on the y-axis, and the four points P whose y-coordinate is ± 5. Write and simplify a Cartesian equation that describes this *hyperbola*. (*Hint*: It may help to start with the equation $PF_1 = PF_2 \pm 8$.) For large values of x and y, the hyperbola is indistinguishable from a pair of lines that intersect at the origin. Find equations for these *asymptotes*.

749. The identities

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \text{and} \quad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

are occasionally useful. Justify them. One method is to use rotation matrices.

750. (Continuation) Compare the graphs $y = \sin 2x$ and $y = 2\sin x\cos x$.

751. The identities

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad \text{and} \quad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

are occasionally useful. Justify them. One method is to use rotation matrices. Another method is to use the established identities for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$.

752. Graph the curve $y = 3\sin x - 4\cos x$, which should look sinusoidal. In other words, it resembles $y = a\sin(x - p)$. Use a calculator to find approximate values for the *amplitude a* and the *phase shift p*.

753. (Continuation) Use the $\sin(\alpha - \beta)$ identity to show that the curve $y = 3\sin x - 4\cos x$ can be rewritten equivalently as $y = a\sin(x - p)$.

754. Asked to simplify the expression $\log_c k + \log_c \dfrac{1}{k}$, Kirby wrote the following:

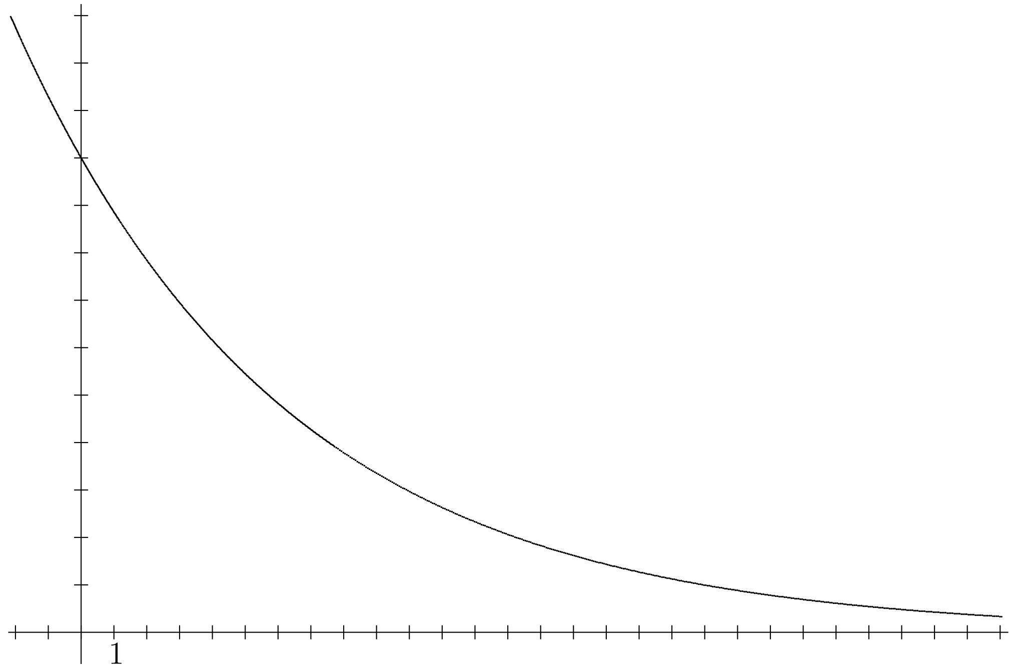
$$\log_c k + \log_c \frac{1}{k} = \frac{\log k}{\log c} + \frac{\log c}{\log k} = (\log k - \log c) + (\log c - \log k) = 0$$

What do you think of Kirby's analysis?

755. Draw the angle described by $\tan^{-1}\frac{12}{5}$. Then make up a geometry problem (and a diagram to go with it) for which the answer is $5\tan\left(\frac{1}{2}\tan^{-1}\frac{12}{5}\right)$.

756. Working in degree mode, find the periods for the three functions $f(x) = \sin(60x)$, $g(x) = \sin(90x)$, and $h(x) = f(x) + g(x)$.

757. The graph $y = ab^t$ of an exponential function is shown below. Using only a pencil and a ruler, calculate the half-life of this function by marking two first-quadrant points on the graph. Repeat the process using a different pair of points. Explain why you and your classmates should arrive at the same answer, no matter what choices you made.



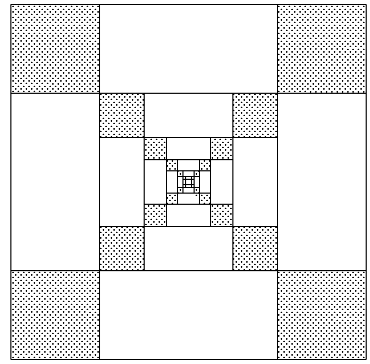
758. If you were to ask for the birthday of a random Exonian, what is the probability that the response will be the 18th of August? (Assume that there are 365 birthdates in a year.) What is the probability that it will *not* be 18 August? If you ask two random Exonians to state their birthdays, what is the probability that *neither* will say 18 August? If you ask all 1017 Exonians this birthday question, what is the probability that *no one* will say 18 August? What is the probability that *someone* (that means at least one person) will say 18 August?

759. Suppose that r is a number strictly between -1 and 1. What can be said about the value of r^n when n is a large positive integer? What can be said about $\dfrac{a - ar^n}{1 - r}$ when n is a large positive integer? Give an example of an infinite geometric series whose sum is $\dfrac{12}{1 - (1/3)}$.

760. In radian mode, what are the ranges of $\sin^{-1}$ and $\cos^{-1}$?

761. If you attempt to "linearize" the equation $y = 5 \cdot 2^x + 8$ by applying logarithms, you will run into difficulty. Why? Explain how the difficulty can be avoided.

762. The figure at right is constructed recursively as follows: Stage 0 consists of a simple uncolored square. Stage 1 is obtained by coloring four congruent squares in the corners, each of which has one sixteenth of the area of the original uncolored square. Stage 2 is obtained by applying the same process to the (smaller) uncolored square left in the center of stage 1. In general, stage n is obtained by applying the corner-coloring process to the uncolored square left in the center of the preceding stage. If the coloring process could be completed, what portion of the original square would be colored?

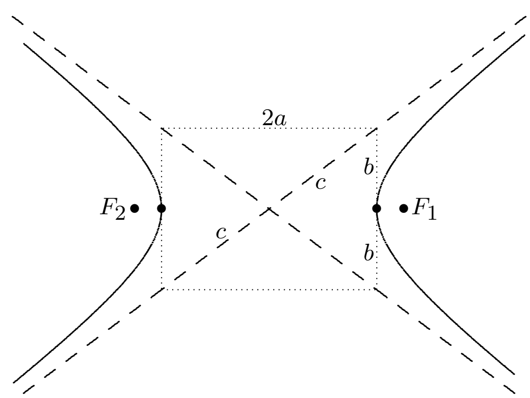


763. A sequence of temperature differences is defined recursively by the initial value $D_0 = 70$ and the equation $D_n - D_{n-1} = -0.2D_{n-1}$. Calculate D_{13}.

764. When $t = 0.2$, the infinite series $1 + t + t^2 + t^3 + \cdots$ *equals* 1.25. What does this mean? For what t-values is it correct to say that the series $1 + t + t^2 + t^3 + \cdots$ has a *sum*? What is the sum?

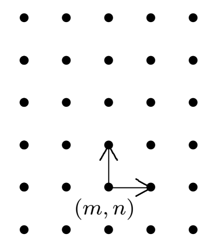
765. Two distinct vertices of a cube are to be randomly chosen. Find the probability that the chosen vertices will be the endpoints of **(a)** an edge of the cube; **(b)** a *face diagonal* of the cube; **(c)** an *interior diagonal* of the cube.

766. *Some hyperbola terminology.* Every hyperbola has two focal points, F_1 and F_2 (which is what Pat and Kim were). The focal points are located on the *major* symmetry axis, with the hyperbola *center* midway between them. The *vertices* are the points where the hyperbola intersects the major axis. As shown, it is customary to let $2a$ be the distance between the vertices and $2c$ be the distance between the focal points. Notice that $a < c$ and that a hyperbola does not intersect its minor axis. According to one definition of hyperbola, the difference between the two focal radii $|PF_1 - PF_2|$ is constant for any point P on the curve. Explain why this constant equals $2a$. (*Hint*: Try a special position for P.)



767. (Continuation) Let $F_1 = (c, 0)$, $F_2 = (-c, 0)$, and $P = (x, y)$. Use the distance formula and some algebra to convert the equation $|PF_1 - PF_2| = 2a$ into the Cartesian form $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2 - a^2} = 1$ Now *define* $b^2 = c^2 - a^2$, find the *asymptotes* of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and confirm that the above diagram is correctly labeled. Notice that the asymptotes are the extended diagonals of the $2a$-by-$2b$ rectangle (shown dotted) that is tangent to the hyperbola at its vertices. It is interesting that the length of a diagonal of this rectangle is exactly $2c$.

768. A bug jumps from lattice point to lattice point on a piece of graph paper, one jump per second, according to the following pattern: From (m, n), the bug jumps only to $(m + 1, n)$ or $(m, n + 1)$, each equally likely. Find all the places the bug could be, two seconds after it leaves the origin $(0, 0)$. Are all these places equally likely?



769. (Continuation) Where could the bug be, three seconds after it leaves the origin? Are all these places equally likely?

770. (Continuation) Expand $(r + u)^2$ and $(r + u)^3$, looking for connections with the jumping bug. In algebra, it is customary to collect terms like rru, rur, and urr into a single term $3r^2u$. Is there any reason to distinguish these terms in this example, however?

771. (Continuation) It would take the bug five seconds to reach $(3, 2)$ from the origin. Given the bug's random behavior, how likely is it that this will happen?

772. An *arithmetic sequence* is a list in which each term is obtained by adding a constant amount to its predecessor. For example, the list 4.0 , 5.2 , 6.4 , 7.6 , . . . is arithmetic. The first term is 4.0; what is the fiftieth? What is the millionth term? What is the n^{th} term?

773. Suppose that $a_1, a_2, a_3, \ldots$ is an arithmetic sequence, in which $a_3 = 19$ and $a_{14} = 96$. Find a_1.

774. Express $0.138918\overline{138918}$ as a geometric series. Find the *rational* sum of the series.

775. Show that the curve $y = 2\sin x - \cos x$ is sinusoidal, by writing it in an equivalent form $y = a\sin(x - p)$.

776. Graph each hyperbola. Plot the focal points and write equations for the asymptotes.
(a) $9y^2 - 4x^2 = 36$ **(b)** $4x^2 - 9y^2 = 36$

777. Let $\mathbf{M} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$. Draw the vectors $\mathbf{v}, \mathbf{Mv}, \mathbf{M}^2\mathbf{v}, \mathbf{M}^3\mathbf{v}, \ldots$ with their tails at the origin. Explain why the heads define a sequence of collinear points. Find the property of matrix $\mathbf{M}$ that is responsible for the collinearity.

778. (Continuation) Show that the distances between consecutive points form a geometric sequence. What is the limiting position of this sequence of points? Suggestion: vectors might prove easier than the Pythagorean Theorem.

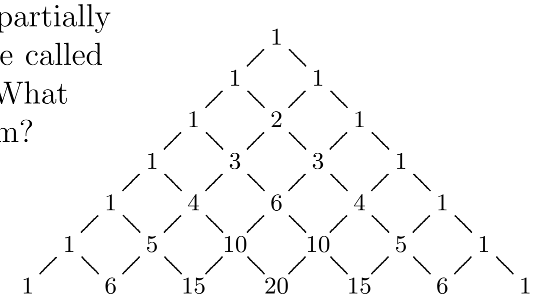
779. The reciprocal of the sine is *cosecant* (abbreviated csc), useful for expressing answers to trigonometry problems without using a division sign. Use this function to express the hypotenuse of a right triangle that has a 12.8-inch side opposite a 25-degree angle.

780. If the cosine of an acute angle is some number k, then what is the *sine* of the same angle? In terms of k, what is the cosecant of this angle? Simplify the expression $\tan(\cos^{-1} k)$.

781. Three distinct vertices of a cube are to be randomly chosen. What is the probability that they will be the vertices of an equilateral triangle?

782. When the *binomial* power $(h+t)^6$ is expanded, the product h^2t^4 appears several times. After like terms are collected, what is the numerical coefficient of h^2t^4? If six pennies are tossed onto the table, what is the probability of seeing two heads and four tails?

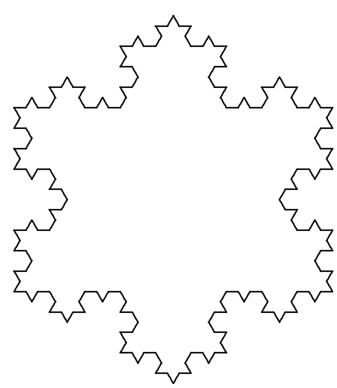
783. *Pascal's Triangle* is an infinite array that is partially shown at right. The numbers that appear in it are called *binomial coefficients*. Explain this terminology. What patterns do you notice, and can you account for them? It is customary to call the top row of this array the 0^{th} *row*. According to this convention, how many entries appear in the n^{th} row, and what is the sum of all the entries in the n^{th} row?



784. Express the repeating decimal $0.5\overline{13}$ as a fraction in lowest terms.

785. Without a graphing tool, graph the following curves, and find equations for their linear asymptotes:
(a) $16x^2 - 9y^2 = 144$ **(b)** $y = 2^x - 3$ **(c)** $x = 2^y - 3$

786. *The Koch snowflake* is an example of a fractal curve of infinite length. As suggested by the figure, however, the area enclosed by this curve is finite. Suppose that the area enclosed by stage 0 (the initial equilateral triangle) is 1. What is the area enclosed by stage 1? by stage 2? by stage n? Show that the area enclosed by the completed snowflake can be obtained with the help of a geometric series.

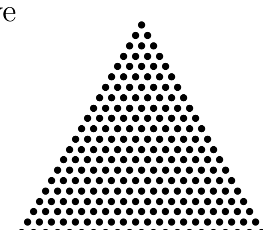


787. Choose a 3-by-3 matrix $\mathbf{M}$ of positive numbers so that the sum of the entries in each column is 1. Choose a column matrix $\mathbf{v}$ whose three positive components sum to 100.
(a) Why is $\mathbf{v}$, $\mathbf{Mv}$, $\mathbf{M}^2\mathbf{v}$, … called a *geometric* sequence? Calculate some of its terms.
(b) If these vectors are drawn with their tails at the origin, the heads define a sequence of coplanar points. Explain why.
(c) What seems to be happening to its terms? Calculate $\mathbf{M}^n$ for some large exponent n.

788. Graph $\sin(x+y) = \frac{1}{2}$ by hand. Check your results with a graphing tool.

789. Using the five digits 7, 7, 8, 8, and 8 once each, how many positive five-digit integers can be formed?

790. *The K. F. Gauss problem.* Find the sum $1+2+3+\cdots+100$.



791. (Continuation) Find sums for the *arithmetic series*:
(a) $156+179+202+\cdots+1996$ **(b)** $1+2+3+\cdots+n$

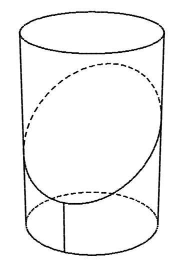
792. The focal points of a hyperbola are $(0, 6)$ and $(0, -6)$, and the point $(5, 6)$ is on one of its *branches*. Find coordinates for the points where the hyperbola intersects its major axis. Also find equations for the asymptotes, and use them to help you draw the curve.

793. (Continuation) The focal points of an ellipse are $(0, 6)$ and $(0, -6)$, and the point $(5, 6)$ is on the ellipse. Find coordinates for the points where the ellipse intersects its major axis.

794. (Continuation) Recall the *reflection property of the ellipse* in order to find the slope of the line λ that is tangent to the ellipse at $(5, 6)$.

795. (Continuation) *Reflection property of the hyperbola.* The line perpendicular to λ at $(5, 6)$ is in fact tangent to the hyperbola. Verify that this is true. Show also that the two tangent lines can be viewed as angle bisectors.

796. Jackie wraps a sheet of paper tightly around a wax candle whose diameter is two inches, then cuts through them both with a sharp knife, making a 45-degree angle with the candle's axis. After unrolling the paper and laying it flat, Jackie sees the wavy curve formed by the cut edge, and wonders whether it can be described mathematically. Sketch this curve, then show that it is sinusoidal. It is very helpful to use radian measure to describe angles in this problem.

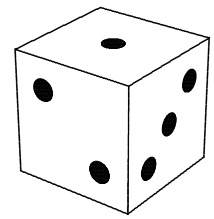


797. *The logistic equation.* Because resources are limited, populations cannot grow forever in an exponential fashion. For example, the wildlife refuge Deer Island can sustain a deer population of only 2000. A quadratic model for the growth of such a population is obtained by assuming that the yearly change in the population is proportional to the size of the population, and also proportional to the difference between the size of the population and its maximum sustainable size. Suppose that the population would grow at an annual rate of 28% if it were not for resource limitations, and that the 240 deer currently on the island represent 12 percent of capacity. The size of the deer population can be predicted recursively by $P_n = P_{n-1} + 0.28P_{n-1} \cdot (1 - P_{n-1})$, where P_n is the size in n years, expressed as a fractional part of capacity. The initial condition is $P_0 = 0.12$.
(a) What will the size of the population be after 1 year? after 10 years? after 20 years?
(b) What if the initial condition were $P_0 = 1.12$? What does this mean?

798. Show that the curve $y = \sin x + 3\cos x$ is sinusoidal, by writing it in an equivalent form $y = a\sin(x + p)$.

799. A long strip of paper, whose thickness is 0.002 inch, is rolled tightly on a cylinder whose radius is 0.75 inch. This produces a cylinder whose radius is 1.75 inches. Estimate the length of the paper strip. What assumptions did you make?

800. Una is going to roll ten standard (six-sided) dice, one after another. What is the probability that
(a) *none* of the dice land showing an ace (a single spot) on top?
(b) *some* (at least one) of the dice land showing an ace on top?
(c) the first die shows an ace on top, but none of the others do?
(d) the last die shows an ace on top, but none of the others do?
(e) *exactly one* of the ten dice shows an ace on top?

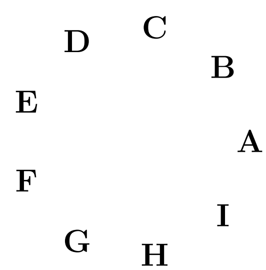


801. The *secant* is the reciprocal of the cosine: $\sec t = \dfrac{1}{\cos t}$. Show how to convert the Pythagorean identity $\cos^2 t + \sin^2 t = 1$ into the form $\sec^2 t - \tan^2 t = 1$.

802. Describe the motion of a particle that is moving along the path described by the equation $(x, y) = (3\sec t, 5\tan t)$, for $0 \le t \le 360$.

803. From a great distance, a hyperbola looks like an "X". Explain. What does a parabola look like from a great distance? Does a parabola have linear asymptotes?

804. How many three-letter words can be formed from the letters A, B, C, D, E, F, G, H, and I, using each letter at most once per word? The words need not *mean* anything, so include GAC and CGA along with BIG and BAD in your count.



805. (Continuation) The points A, B, C, D, E, F, G, H, and I shown at right are vertices of a regular polygon. If three of the points are chosen, a triangle is determined. How many triangles can be formed in this way?

806. (Continuation) What is the probability that a randomly chosen triangle will be equilateral? What is the probability that a randomly chosen triangle will be a right triangle?

807. By means of a formula, invent an example of a periodic but *non-sinusoidal* function f, whose period is 50, and whose values $f(x)$ oscillate between the extremes -3 and 7.

808. Find parametric equations for the graph of $4x^2 - 9y^2 = 144$.

809. There are many hyperbolas whose asymptotes are $y = \pm\frac{2}{3}x$. Sketch and write an equation for such a hyperbola, given that its vertices are
(a) $(-6, 0)$ and $(6, 0)$; **(b)** $(0, -5)$ and $(0, 5)$.

810. Choose two first-quadrant points on the curve $y = \sqrt{4x^2 + 9}$, both far from the origin. Calculate the slope of the line they determine. Could you have anticipated the answer?

811. Let $f(x) = \sqrt{9 - x^2}$ for $-3 \le x \le 3$. Sketch the graph $y = f(x)$, then compare that graph with the graphs of the following related functions. Be prepared to discuss the role of the *parameter* 2 in each; in particular, how does it affect the domain and the range?
(a) $y = 2f(x)$ **(b)** $y = f(2x)$ **(c)** $y = f(x) + 2$ **(d)** $y = f(x + 2)$

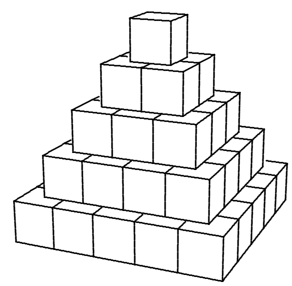
812. Starting at the origin, a bug jumps randomly along a number line. Each second, it jumps one unit to the right or one unit to the left, either move being equally likely. Describe all the places that the bug could be after eight seconds, and tell how likely each of them is.

813. (Continuation) Describe all the places that the bug could be after nine seconds, and tell how likely each of them is.

814. Let $F = (0, 9)$ be the focus and the line $y = 1$ be the directrix. Plot several points P that are three times as far from the focus as they are from the directrix, including the vertices on the y-axis. The configuration of all such P is a hyperbola of eccentricity 3. Use the distance formula to write an equation for the hyperbola. Find the values of a, b, and c for this curve, then calculate the ratio c/a. Is the result what you expected?

815. Your experiments with sugar-cube pyramids led to series of consecutive squares, which look like $1 + 4 + 9 + \cdots + n^2$. This series is neither arithmetic nor geometric, thus a concise formula for its sum $S(n)$ is not readily available. After you explain why the series is neither arithmetic nor geometric, calculate the five specific values $S(1)$, $S(2)$, $S(3)$, $S(4)$, and $S(5)$.

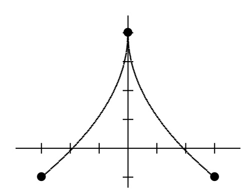
816. (Continuation) Sasha builds a sugar-cube pyramid by stacking centered square layers. The dimension of each layer is one less than the dimension of the layer immediately below it. The bottom layer is n-by-n. Sasha would like a formula for the total number of sugar cubes in such a pyramid. Sasha knows the formula $n(n+1)/2$ for the sum of consecutive integers $1+2+3+\cdots+n$. Because $n(n+1)/2$ is a quadratic function of n (it can be written in the form an^2+bn+c), Sasha guesses that the formula for $1+4+9+\cdots+n^2$ is a cubic function of n. In other words, $S(n)$ can be written $an^3 + bn^2 + cn + d$. Use the data $S(1) = 1$, $S(2) = 5$, $S(3) = 14$, and $S(4) = 30$ to determine values for a, b, c, and d. Test your formula on $S(5)$. Use the formula to express the volume of the sugar-cube pyramid as a fractional part of the volume of an n-by-n-by-n cube. What is this ratio when n is a very large number?



817. Express the geometric series $28 + 16.8 + \cdots + 28(0.6)^{15}$ in sigma notation. Then find the sum of these sixteen terms.

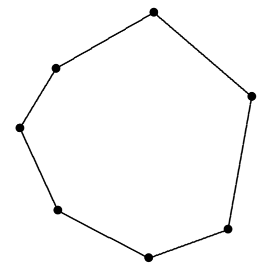
818. The graph of $y = g(x)$ for $-3 \le x \le 3$ is shown at right. Notice that $(0, 4)$, $(3, -1)$, and $(-3, -1)$ are points on the graph. Sketch the graph of each of the following related functions. Be prepared to discuss the role of the parameter 2 for each function. Write the domain and range of the functions.

(a) $y = 2g(x)$ **(b)** $y = g(2x)$ **(c)** $y = g(x) + 2$ **(d)** $y = g(x + 2)$



819. Show that csc and cot (the reciprocals of sin and tan) can be used to parametrize a hyperbola, just as sec and tan can.

820. Interpreted in degree mode, the equation $y = \cos(72000t)\sin(2x)$ models the motion of a stretched string that is 180 centimeters long ($x = 0$ is one end of the string and $x = 180$ is the other), and that is vibrating 200 times per second.
(a) There are times when the model predicts that the string is straight. Find two consecutive such times.
(b) The center of the string never moves. Find a point on the string that is moving the fastest at an instant when the string appears straight. How fast is this point moving?



821. How many different labeled versions of the heptagon shown at right could you create by applying the seven letters T, H, R, E, A, D, and S (in any order) to the seven vertices?

822. Write an equation for the sinusoidal curve that has a crest at the point $(50, 7)$ and an adjacent valley at $(150, -7)$.

823. Write equations for at least two sinusoidal curves that have a crest at the point $(150, 7)$ and a valley at $(200, -7)$.

824. Write an equation for the hyperbola whose vertices are $(1, 5)$ and $(1, -1)$, and whose focal points are $(1, 7)$ and $(1, -3)$. Draw a graph and find equations for the asymptotes.

825. Without using a calculator to expand $(a + n)^5$, find the coefficient of its a^2n^3 term.

826. (Continuation) What is the coefficient of the a^2 term in the expansion of $(a - 2)^5$?

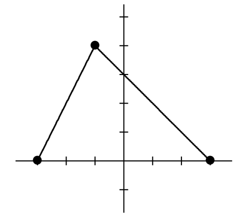
827. Five standard (six-sided) dice are rolled, one at a time. What is the probability that
(a) the *first two* dice show aces, and the next three do not?
(b) two of the five dice show aces, and the other three do not?

828. Given that $P(x) = 3960(1.06)^x$, find a formula for the *inverse function* P^{-1}. In particular, calculate $P^{-1}(5280)$, and invent a context for this question. Graph $y = P^{-1}(x)$.

829. A fact from physics: The time required to fall from a height of h feet (or to rise to that height after a bounce) is $\sqrt{h}/4$ seconds. Suppose that a ball, whose rebound ratio is 64 percent, is dropped from a height of 25 feet.
(a) When the ball strikes the ground for the second time, it will have traveled 57 feet in total. Confirm that this is a true statement.
(b) How much time passes between the initial drop and the second impact?
(c) Avery responds to the preceding question, "That's easy, you just divide $\sqrt{57}$ by 4 and get 1.89 seconds." Brooks responds, "I think you meant to divide $\sqrt{25 + 16}$ by 4 and get 1.60 seconds." What do you think of these remarks, and why?

830. (Continuation) How much time passes between the initial drop and the hundredth impact? How far has the ball traveled by then? How far does the ball travel if it is left to bounce "forever"? How much time does all this bouncing actually take?

831. A graph of $y = h(x)$ for $-3 \le x \le 3$ is shown. For each of the following related functions, sketch a graph, explain the role of the numerical parameter, and write the domain and range.
(a) $y = -4h(x)$ **(b)** $y = h(x/2)$ **(c)** $y = h(x) - 1$ **(d)** $y = h(x - 3)$



832. What do the functions $f(x) = \log(2 + \sin 60x)$, $g(x) = 2^{\sin 60x}$, and $h(x) = |2 + 3\sin 60x|$ have in common? How do they differ?

833. How many nine-letter words can be formed from the letters of *hyperbola*, using each letter once per word? The words do not have to actually spell anything, of course.

834. The product $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ is usually abbreviated by 9!, and read *9 factorial.* Because factorials are often large, they are a challenge to compute and display. Consider 1000!, for example. What does the information $\log(1000!) = 2567.604644$ tell us about this gigantic number? Invent a question that has 1000! as its answer.

835. What is the probability that the thirty-two residents of Lamont have thirty-two different birthdays? What is the probability that there is at least one birthday coincidence in Lamont?

836. If five standard (six-sided) dice are tossed onto the table, what is the probability that
(a) all of them will show an odd number on top?
(b) no aces or deuces (that means ones or twos) will show on top?
(c) the five dice will show five different values on top?

837. To convert Celsius temperatures to Fahrenheit temperatures, the function F defined by $F(x) = 32 + 1.8x$ is useful. Confirm this, then find a formula for the function C (which could also be called F^{-1}) that converts Fahrenheit temperatures to Celsius temperatures.

838. To graph the equation $x = 2^y - 3$, one approach is to rewrite the equation so that y is expressed as a function of x. Do so.

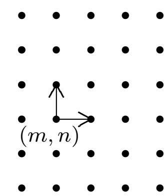
839. A superball is dropped from a height of h feet, and left to bounce forever. The rebound ratio of the ball is r. In terms of r and h, find formulas for
(a) the total distance traveled by the ball;
(b) the total time needed for all this bouncing to take place.

840. How many four-letter words can be formed by using the letters in *facetious*? There are nine letters available, and each one can be used at most once per word.

841. A large wooden cube is formed by gluing together 1000 small congruent cubes, and then it is painted red. After the paint is dry, the large cube is taken apart into small cubes again. How many of these small cubes have paint on three of their faces? on exactly two of their faces? on exactly one of their faces? on none of their faces?

842. (Continuation) Here is a curiosity: Expand the binomial power $(a+b)^3$ and then replace a by 2 and replace b by 8.

843. A bug jumps from lattice point to lattice point on a piece of graph paper, one jump per second, as follows: From (m, n), there is a 60 percent chance that the bug jumps to $(m + 1, n)$ and a 40 percent chance that it jumps to $(m, n+1)$. Find all the places the bug could be, two seconds after it leaves the origin $(0, 0)$. Are they equally likely?



844. (Continuation) It would take the bug five seconds to reach $(3, 2)$ from the origin. How likely is it that this will actually happen?

845. A hyperbola equation can be written in *factored form*, as in $(4x + 3y)(4x - 3y) = 72$. This enables the *asymptotes* to be written down easily: $4x + 3y = 0$ and $4x - 3y = 0$. What is the reasoning behind this statement? Apply this reasoning to sketch the following graphs (all actually hyperbolas). Draw the asymptotes first, then plot one convenient point (an axis intercept, for example), then use symmetry to freehand the rest of a rough sketch.

(a) $(x + 2y)(x - 2y) = 36$ **(b)** $(x + 2y)(x - 2y) = -36$
(c) $(x + 3y)(x - y) = 9$ **(d)** $xy = 18$

846. (Continuation) To find coordinates for the vertices of examples (c) and (d), a special approach is needed. One familiar method uses angle bisectors. After you find their coordinates, add the vertices to your sketches.

847. Show that the ellipse $16x^2 + 25y^2 = 400$ and the hyperbola $400x^2 - 225y^2 = 1296$ are *confocal*. Find coordinates for the four points where the graphs intersect each other. Then find the size of the *angle of intersection* at one of these points, to the nearest $0.1°$.

848. Find periods for $f(x) = \sin(7.2x)$, $g(x) = \sin(4.5x)$, and $h(x) = f(x) + g(x)$.

849. The domain of function f is all numbers between -4 and 6, inclusive, and the range of f is all numbers between -3 and 5, inclusive. Find the domain and range of $g(x) = 3f(x/2)$.

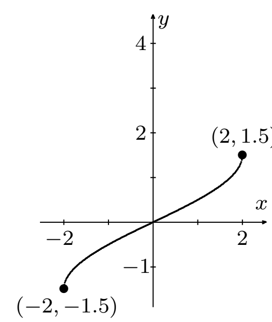
850. Pascal's Triangle is associated with binomial situations (right versus left, heads versus tails, etc). Its ninth row is 1 9 36 84 126 126 84 36 9 1. The zeroth and ninth entries in this row are both 1. Notice that the third entry in this row can be calculated in the form $\dfrac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1}$. Find analogous presentations for the other entries in this row.

851. (Continuation) Recall #805, about counting the number of triangles that can be formed by choosing three dots from nine available dots. *This is a binomial question.* Explain why. This justifies the appearance of the answer in Pascal's Triangle. The third entry in the ninth row is usually denoted by ${}_9C_3$ or by $\binom{9}{3}$, and read "nine choose three."

852. *Repaying loans, part I.* The bank has just granted Jordan a \$10 000 loan, which will be paid back in 48 equal monthly installments, each of which includes a 1 percent charge on the unpaid balance. The problem is to calculate the correct monthly payment.
(a) Suppose that the monthly payments are all \$150. The first payment must include \$100 just for interest on the \$10 000 owed. The other \$50 reduces the debt. That leaves a debt of \$9950 after the first payment. Let A_n be the amount owed after n payments (so that $A_0 = 10\,000$). Justify the recursion $A_n = 1.01A_{n-1} - 150$, then apply it repeatedly to show that Jordan still owes \$6938.87 after 48 payments of \$150 have been made.
(b) Because $P = 150$ is not large enough to pay off the loan in 48 installments, try the following set of values of P: 200, 250, 300, 350. For each value of P, apply the recursion $A_n = 1.01A_{n-1} - P$ with $A_0 = 10\,000$ until you have A_{48}.
(c) Plot the five points (P, A_{48}) on a graph, one of which is $(150, 6938.87)$. You should notice that these five points are collinear.
(d) Find an equation for the line containing the points from (c). Use this line to find the payment P that makes $A_{48} = 0$. This is the payment that pays off the loan.

853. In radian mode, graph both $y = \sin^{-1} x$ and $y = \pi - \sin^{-1} x$ on the same coordinate-axis system. What do you notice?

854. The diagram shows a graph $y = f(x)$. Add the graph of $y = 3 - f(x)$ to the diagram.



855. Verify that the line $8y - x = 4$ is tangent to the hyperbola $48y^2 - x^2 = 48$ at $P = (12, 2)$ and that the focal points for this hyperbola are $F_1 = (0, -7)$ and $F_2 = (0, 7)$. Calculate the size of the two acute angles formed by the tangent line and the focal radii.

856. (Continuation) Sketch the hyperbola $48y^2 - x^2 = 48$ and the ellipse $4x^2 + 3y^2 = 588$. Verify that they are confocal and share point P. The two tangent lines to the curves at point P are related. Sketch the tangent lines and explain the relationship. Note that you have encountered the equations of *both* tangent lines already.

857. When the binomial power $\left(\frac{5}{6} + \frac{1}{6}\right)^4$ is expanded, five terms should appear. What are they, and what does each one mean?

858. The Metropolis Cab Company, which has a fleet of 1000 taxis, has divided the city into three zones, with one office in each zone. The matrix shows the distribution of destinations from each zone. For example, 23% of the pickups in zone 1 are discharged in zone 3. After discharging a fare, an MCC taxi waits in that zone for a new fare. On the basis of this long-term data, what do you expect that the current distribution of MCC taxis is?

$$\begin{bmatrix} 0.42 & 0.33 & 0.12 \\ 0.35 & 0.48 & 0.31 \\ 0.23 & 0.19 & 0.57 \end{bmatrix}$$

859. Find the asymptotes, vertices, focal points, and eccentricity for the hyperbola $xy = 16$.

860. Using the ten letters p, p, p, q, q, q, q, q, q, q only, how many ten-letter words can be formed? What is the coefficient of p^3q^7 when the binomial $(p+q)^{10}$ is expanded?

861. Compare the domains and ranges of the functions $f(x) = 2\log x$ and $g(x) = \log(x^2)$.

862. Find a simple expression for each of the sums: **(a)** $\displaystyle\sum_{n=1}^{96} \log n$ **(b)** $\displaystyle\sum_{n=0}^{\infty} \sqrt{0.9^n}$

863. Explain why there are $9 \cdot 8 \cdot 7 \cdot 6$ different four-letter words that can be formed by using only the letters of *logarithm*. This product is often denoted ${}_9P_4$, and read "nine permute four." A *permutation* is an arrangement of things (letters, for example).

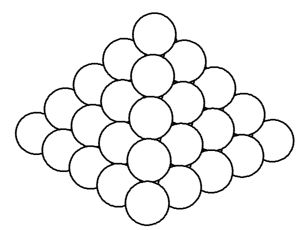
864. Find a parametric description for the hyperbola $\dfrac{(x-3)^2}{16} - \dfrac{(y+2)^2}{25} = 1$.

865. In how many ways can a mathematics class of twelve students and one teacher seat itself at a circular Harkness table?

866. When you listen to the sound of a bouncing ping-pong ball that has been dropped onto a cement floor, what mathematical pattern do you hear?

867. Given a geometric sequence x_0, x_1, x_2, $\ldots$, form the sequence of gaps y_1, y_2, y_3, $\ldots$, where $y_n = x_n - x_{n-1}$. Show that the y-sequence is also geometric. What is its ratio?

868. What are the domain and range of the function $\tan^{-1}$?



869. A grocer has 1015 spherical grapefruit, which are to be stacked in a square pyramid — one in the top layer, four in the next layer, etc. How many layers will the completed pyramid have? The diameter of each grapefruit is 6 inches. Find the height of the completed pyramid.

870. Given the information $-1 < \log_4 p < 1.5$, what can you say about the possible values of p?

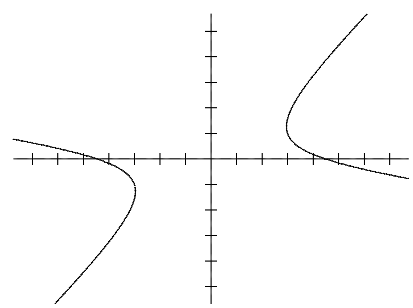
871. Rewrite the equation $\log y = b + m \cdot \log x$ in a form that makes no reference to logarithms.

872. *Repaying loans, part II.* The bank has just granted Jordan a \$10 000 loan, which will be paid back in 48 equal monthly installments, each of which includes a 1 percent interest charge on the unpaid balance. Previously, Jordan has solved the loan payment problems through a guess-and-check procedure involving a linear equation. Now Jordan, who knows about geometric series, would like to develop a general formula for the correct loan payment.
(a) Let A_n be the amount owed after n payments (so that $A_0 = 10\,000$), let $r = 0.01$ be the monthly interest rate, and let P be the monthly payment (which might not be 300). Explain why $A_1 = A_0 - (P - rA_0) = (1 + r)A_0 - P$, and $A_2 = (1 + r)A_1 - P$, then write a recursive equation that expresses A_n in terms of A_{n-1}.
(b) Apply the recursive equation to express A_n in therms of A_0. For example, you can write $A_2 = (1 + r)A_1 - P = (1 + r)[(1 + r)A_0 - P] - P$, which can be reorganized as $A_2 = (1 + r)^2 A_0 - (1 + r)P - P$. You should see a pattern developing! It involved the finite geometric series $P + (1 + r)P + (1 + r)^2 P + ... + (1 + r)^{n-1}P$.
(c) Explain why $A_{48} = 0$ for Jordan's loan Then set $A_{48} = 0$ in your answer to (c), and solve for P. This expresses the monthly payment in terms of A_0 (which is 10 000), r (which is 0.01), and n (which is 48). In Jordan's case, the monthly payment is less than \$300.

873. The twenty-person Mathematics Department is forming a four-person committee to draft a technology proposal. How many possible committees are there? How many include your mathematics teacher? How many do *not* include your mathematics teacher? Hmm . . .

874. What is the difference between the mathematical uses of the words *series* and *sequence*?

875. In how many of the 9! permutations of *facetious* do the vowels occur in alphabetic order? If you get stuck on this one, you can test your approach by trying it on an easier version of the question — the 4! permutations of *face*.



876. The figure at right shows part of the graph of $x^2+6xy-7y^2 = 20$. Identify the asymptotes of this hyperbola, then use symmetry to help you find coordinates for the vertices, which are the points closest to the origin.

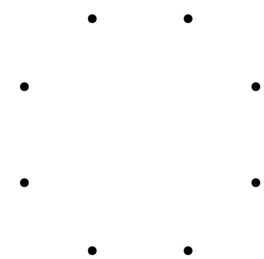
877. (Continuation) A similar approach to the one used in #485 can be used to find a parametric description of this hyperbola.
(a) Calculated the distance between the vertices and the distance between the foci.
(b) Find parametric equations and sketch the graph using a graphing tool.

878. The smallest unit of information is called a *bit*. A sequence of eight bits is called a *byte*. A bit represents two possible values (say 0 or 1). How many different values can a byte represent?

879. We have seen that ${}_9C_3$ can be calculated as $\dfrac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1}$. Show that this can be expressed more compactly as $\dfrac{9!}{3!\,6!}$. Write a general formula for ${}_nC_r$ in terms of factorials.

880. Simplify the ratios: **(a)** $\dfrac{20!}{18!}$ **(b)** $\dfrac{\log 32}{\log 8}$ **(c)** $\dfrac{\log (m^5)}{\log (m^3)}$ **(d)** $\dfrac{1+r+r^2+\cdots+r^{39}}{1+r+r^2+\cdots+r^{19}}$

881. Choose three of the eight dots shown in the figure at right; they determine a triangle. There are 56 such triangles possible. How many quadrilaterals can be obtained by choosing four dots? How many pentagons are possible? There are several questions of this sort that can be asked about these eight dots. Where are the answers to all these questions found?



882. The function E defined by $E(x) = 2^{\sin x}$ is periodic, but not sinusoidal. Explain.

883. Consider the two points $D = (2, -3)$ and $C = (1, 4)$. Point $P = (p, 0)$ is on the x-axis.
(a) Write a formula for the difference of the lengths PC and PD.
(b) Show that this difference is a maximum when $p = 5$.

884. (Continuation) Consider the reflection of D over the x-axis onto D'.
(a) The line CD' crosses the x-axis at point W, find the coordinates of W.
(b) How does W relate to P?
(c) What can you say about the relationship between the angle that line CW makes with the x-axis and the angle that the line DW makes with the x-axis?

885. What is the probability that all the members of your mathematics class have different birthdays? What is the probability that there is a birthday coincidence in the group?

886. (Continuation) How large a group is needed, in order that there be at least a 50 percent chance of finding at least one birthday coincidence somewhere in the group?

887. If a hundred pennies are tossed onto the floor, the *most likely* outcome is that fifty of them land heads and fifty land tails. What is the probability of this actually happening, however?

888. A *proportion* is created by filling the four blank spaces of ___ : ___ = ___ : ___ with numbers. Given four distinct numbers a, b, c, and d, how many ways are there of using all four numbers to fill the blanks? Assuming that at least one of these permutations does produce a correct proportion, what is the probability that a *random* permutation of a, b, c, and d will produce a correct proportion?

889. A regular triangular pyramid (a *regular tetrahedron*, that is) is sliced by four cutting planes, each of which is parallel to a face of the pyramid and bisects the altitude drawn to that face. This dissects the pyramid into five pieces, four of which are smaller pyramids. Describe the fifth piece, name it, and find its volume.

890. Simplify the expression $\dfrac{a \cdot r^{n+1} - a \cdot r^n}{a \cdot r^n - a \cdot r^{n-1}}$, then explain the significance of the result.

891. What is the monthly payment needed to repay a \$50000 loan in ten years, if the bank charges 0.8 percent per month?

892. Another common name for the inverse sine function is *arcsin*, which is an abbreviation of *find the arc whose sine is*. Explain this terminology.

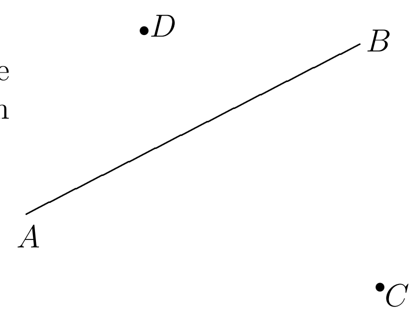
893. From the thirteen available Preps, the P.E. teacher is about to choose five to form a basketball team. There are $1287 = {}_{13}C_5$ possible teams. How many of these teams include KC, who is the tallest in the group? How many of the teams do not include KC? The answers to these questions illustrate a familiar property of Pascal's triangle — what is it?

894. In December Corey took out a 1000-dollar loan with monthly interest rate 0.7 percent. In order to pay back the loan, Corey has been paying \$87.17 a month since January. Explain why the sequence defined recursively by $x_0 = 1000$ and $x_n = 1.007x_{n-1} - 87.17$ for positive n describes the balance of Corey's debt throughout the repayment of the loan. How many payments does Corey need to pay everything back? How much does this loan cost? What would the monthly payment have been if Corey had been scheduled to pay back the loan in twenty-four months?

895. What is the probability that this year's graduation will fall on the birthday of *exactly one* of the 336 Seniors? What is the probability that there is *more than one* such Senior?

896. What is the domain of the function f defined by $f(x) = \sin(\cos^{-1} x)$? What is its range of values? Find an equivalent way to describe the graph $y = f(x)$.

897. For the first 31 days of your new job, your boss offers you two salary options. The first option pays you \$1000 on the first day, \$2000 on the second day, \$3000 on the third day, and so forth — \$1000n on the n^{th} day, in other words. The second option pays you one penny on the first day, two pennies on the second day, four pennies on the third day — the amount doubling from one day to the next. Let A_n be the total number of dollars earned in n days under the first plan, and G_n be the total number of dollars earned in n days under the second plan. Write recursive and explicit descriptions of both A_n and G_n. Which plan is preferable? Explain.



898. Suppose that P is the point on line AB that makes the difference of distances $|CP - PD|$ as large as possible. Explain why the angles CPB and DPB must be the same size.

899. *Reflection property of the hyperbola.* Suppose that P is a point on a hyperbola whose focal points are F_1 and F_2. Draw the intersecting lines PF_1 and PF_2, as well as the bisectors of the four angles they form. This problem is about the bisector that separates F_1 and F_2. Prove the following: Given any point Q *other than* P on this line, the difference $|QF_1 - QF_2|$ is smaller than the difference $|PF_1 - PF_2|$. Conclude that the line intersects the hyperbola only at P.

900. Work in *radian* mode for this one. The line $y = 0.5x$ intersects the sine curve in three places. What are the coordinates of these intersection points? What would happen if the graphs were drawn in degree mode?

901. (Continuation) Again working in radian mode, draw the graph of $y = \sin x$. Then consider all straight lines of *positive* slope that can be drawn through the origin. If one of these lines is randomly chosen, what is the probability that it will intersect the sine graph more than once?

902. The point $P = (6, 5)$ is on the hyperbola $5x^2 - 4y^2 = 80$. Verify this and make a sketch. Then find an equation for the line that intersects the hyperbola tangentially at P.

903. There are 26! permutations of the word *abcdefghijklmnopqrstuvwxyz*. At two permutations per line and 60 lines per page and 12 pages per minute, how much time will be needed to print them all?

904. Sasha takes out a \$250 000 mortgage loan. The bank charges 0.75 percent monthly interest on the unpaid balance. Calculate the monthly payments and the total amount that Sasha will pay the bank, assuming that the loan duration is **(a)** 30 years; **(b)** 15 years.

905. Tyler buys a new laptop for \$1200, and pays for it using a credit card. This particular card charges 1.5 percent monthly interest on the unpaid balance, and requires a minimum payment of \$20 each month. Suppose that Tyler pays only the minimum amount each month. How long will it take Tyler to pay off the debt in this way, and how much will the bank eventually receive for its \$1200 loan to Tyler?

906. Figure out as much as you can about the planes $2x + 3y + 6z = 12$, $2x + 3y + 6z = 32$, and $2x + 3y + 6z = 52$. Justify your conclusions.

907. A lottery winner is given two payment options: Receive 131 million dollars in 25 yearly installments of equal size, the first payable immediately, or receive a single immediate payment of 70.3 million dollars. Assuming that these plans are of equal value to the state lottery system, what interest rate is the state getting on its investments?

908. Let $f(x) = \cos^{-1}(\cos x)$, which is valid for all x. Sketch the graph of this periodic function f. Find its period and the range of its values $f(x)$.

909. Simplify $(\log_a b)(\log_b a)$.

910. Sasha tried to graph both $y = \sin x$ and $y = \sin^{-1} x$ on the same coordinate-axis system, using degree mode for the calculations. Sasha found this exercise to be difficult and confusing. Explain why it is clearer to use radian mode. How large a graphing window is needed to see all the necessary detail?

911. Working in radian mode, graph both $y = \sin^{-1} x$ and $y = \frac{\pi}{2} - \sin^{-1} x$ on the same coordinate-axis system. What is the customary name for the second function?

912. As of 11 December 2003, the largest integer known to be prime was $2^{20996011} - 1$. It was found as part of the GIMPS (Great Internet Mersenne Prime Search), which harnessed the power of 211000 personal computers during the eight-year project. It was reported in the *Boston Globe* that this new prime is a 6320430-digit number. Make calculations to confirm this statement. Elsewhere in the article, it was also reported that it would take a book of about 1500 pages to print all the digits. Make calculations to show that this is also a plausible statement. The title of the article is "Largest prime number discovered." Does this make sense? Explain. Does this prime still hold the record?

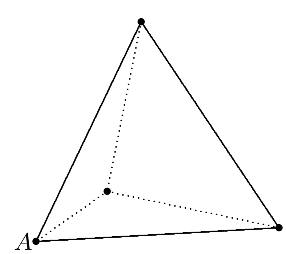
913. The graph of $481x^2 - 384xy + 369y^2 = 5625$ is an ellipse, but the presence of the xy term in the equation prevents us from recognizing the curve. One approach to this problem is to use a rotation of coordinates, as follows:
(a) Replace each x in the equation by $0.6X - 0.8Y$ and each y by $0.8X + 0.6Y$. Verify that the coefficient of XY in the resulting equation is 0.
(b) Find the dimensions of the ellipse that is represented by the XY equation.
(c) The coordinates (X, Y) were obtained by applying a clockwise rotation to (x, y). Calculate the size of this acute angle.
(d) In the xy-coordinate system, the graph of the given equation is an ellipse whose major symmetry axis is a line of positive slope through the origin. Use the preceding information to find the slope of this line and coordinates for the vertices of the ellipse. Sketch the curve.

914. The graph of $3x^2 + 5xy - 2y^2 = 12$ is a hyperbola, whose major symmetry axis is the line $y = (\tan 22.5)x$. Make calculations that confirm this statement.

915. A *nautical mile* was once defined as the distance represented by one minute of latitude. Find out what the current definition is, and why the original definition was changed.

916. Eugene has an object worth \$75000, which is expected to increase in value by \$5000 per year. Furthermore, Eugene anticipates that money invested today will earn 5% interest annually for the foreseeable future.
(a) If Eugene sells the object a year from now, what is its *present value*?
(b) What is the present value of the object if the sale occurs two years from now? twenty years from now?
(c) In how many years should Eugene sell the object to maximize its worth?

917. Let A, B, C, and D be the vertices of a regular tetrahedron, each of whose edges is 1 meter long. A bug starts at vertex A and crawls along the edges of the tetrahedron. At each vertex, it randomly chooses which of the three available edges it will follow next, each of the three being equally likely. What is the probability that the bug will find itself at vertex A after it has crawled 7 meters?



918. *Repaying loans, part III.* Jordan is in the market for a house. He is looking to buy a \$250 000 home. To buy the house, Jordan will make a down payment of \$50 000 and take out a \$200 000 mortgage to finance the purchase. He will pay off the mortgage in 360 equal monthly payments.
(a) Jordan is shopping around for a loan and sees various banks advertising a range of interest rates. Using annual interest rates of 3%, 3.5%, 4%, ... 6%, determine the monthly payment needed to pay off the mortgage (banks divide the annual interest rate by 12 to find the monthly rate). Also, for each interest rate calculate the total amount paid during the lifetime of the loan. What is the effect of the relatively small changes to the interest rate?
(b) Jordan qualifies for a 30-year fixed rate mortgage at 4% annual interest. Create a spreadsheet with the following columns: month (1-360), remaining loan balance, interest paid this month, principal paid this month. Make one scatter plot with month on the horizontal axis; interest paid this month and principal paid this month should both be on the vertical axis. After which month does over half your payment go to principal? Would this remain the same if the annual interest rate was 9%?
(c) Many financial advisers encourage clients to pay an additional \$100 above their monthly mortgage payment. Investigate the effect of Jordan adding \$100 above his monthly payment on his 4% loan.
(d) Suppose Jordan had an option of a 15-year fixed mortgage versus a 30-year fixed mortgage. Compare the effects on the monthly payment and total payments of each loan.

919. Draw a triangle that has two 72-degree angles, and two 2-inch sides. Bisect one of the 72-degree angles. This creates two isosceles triangles, one of which is similar to the triangle you started with. The process can thus be repeated forever, each time applied to a triangle with two 72-degree angles. Make a list of the lengths of the *longest sides* of this sequence of triangles. What kind of numerical sequence do you discover?

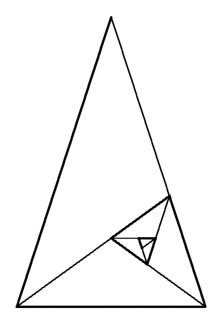
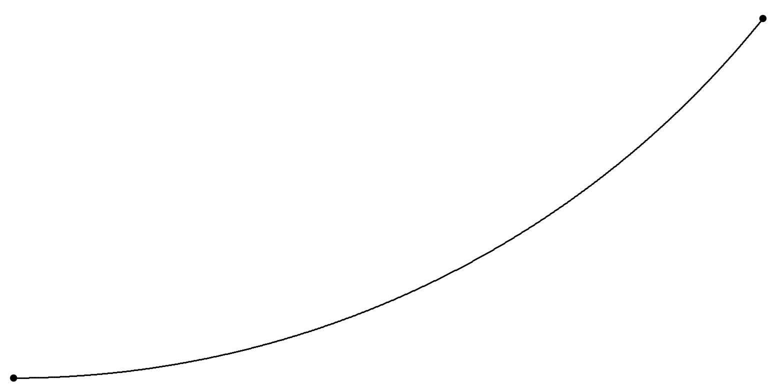


Diagram to accompany #428



amplitude: See *sinusoidal*.

angles between curves: If two curves intersect at point P, the angles between these curves at point P are the angles formed by the intersection of the two lines tangent to the curves at point P. Note that two pairs of vertical angles are formed by the intersecting tangent lines. [203]

angular size of an arc: This is the size of the *central angle* formed by the radii that meet the endpoints of the arc. [44, 371]

apparent size: Given an object, its apparent size is the size of the *angle* subtended by the object at the viewer's eye. For example, the apparent size of the moon is about 0.5 degree for any viewer on Earth. [12, 32]

area of a sector: This is half the product of its arclength and its radius. [31,51,90]

area of a sphere: The surface area of a sphere of radius r is $4\pi r^2$, which (as Archimedes showed long ago) is two thirds of the surface area of the circumscribed cylinder.[304, 337, 379]

area of a triangle: This can be calculated from SAS information by $\frac{1}{2}\,ab\sin C$. [82, 110]

arithmetic sequence: A list in which each term is obtained by adding a constant amount to the preceding term. [625, 740, 772]

arithmetic series: The sum of an arithmetic sequence. To evaluate such a sum, you can simply average the first and last terms, then multiply by the number of terms. [791]

associative property: For addition, this is $(a+b)+c = a+(b+c)$, and for multiplication, it is $(a\cdot b)\cdot c = a\cdot(b\cdot c)$. This property also applies to vectors and matrices. [712]

astronomical unit: The mean distance from the earth to the sun, 93 million miles, is a useful unit for expressing distances in our solar system. [433]

asymptote: Two graphs are asymptotic if they become indistinguishable as the plotted points get farther from the origin. [310, 400, 739, 748, 759, 767]

average speed: The average speed during a time interval is $\dfrac{\text{total distance}}{\text{total time}}$. [512]

binomial coefficients: These appear when a binomial power $(x+y)^n$ is expanded. For example, when $(x+y)^5$ is expanded, the coefficient of the term xy^4 is 5 and the coefficient of the term x^2y^3 is 10. [770, 783]

branch: One of the two connected pieces of a *hyperbola*. [792]

Cartesian form: An equation written in terms of the rectangular coordinates x and y. For example, $x^2 + y^2 = 784$ is a Cartesian equation for a circle. [331, 443]

centroid: The point whose coordinates are obtained by averaging the coordinates of a given set of points. [340]

central angle: An angle formed by two radii of a circle. [186]

circle equation: A circle with center at (a, b) and radius r can be described by the equation $(x - a)^2 + (y - b)^2 = r^2$. [41]

circle equation (parametric): A circle with center at (a, b) and radius r can be described by the system of parametric equations $x = a + r\cos t$ and $y = b + r\sin t$. [45, 64,77]

circular functions: This describes *cosine* and *sine*, for they are coordinates of a point on the unit circle: If $A = (1, 0)$ and arc AP has angular size θ, then $P = (\cos\theta, \sin\theta)$. These functions obey the Pythagorean rule $(\cos\theta)^2 + (\sin\theta)^2 = 1$ for every θ. [30, 53]

circumcenter: The perpendicular bisectors of the sides of a triangle are concurrent at this point, which is equidistant from the vertices of the triangle. [122]

coefficient matrix: A rectangular array of coefficients extracted from a system of linear equations. [177, 263]

column vector: A single column of a matrix can be thought of as a vector. [144]

combination: An *unordered* collection of things, typically chosen from a larger collection. There are ${}_nC_r = n(n-1)\cdots(n+1-r)/r!$ ways to choose r things from n things. [851]

commutative property: For addition, this is $a + b = b + a$, and for multiplication it is $a \cdot b = b \cdot a$. This property does not apply to matrix multiplication. [269,309]

compound interest: When interest is not withdrawn from the bank, the additional money in the account itself earns interest. [417]

concentric: Curves are called *concentric* if they have a common center. [250,596]

cone: A surface that is formed by joining all the points of a *base* circle to a *vertex*. If the vertex is closer to the center of the circle than it is to any other point of the base plane, the cone is called *right*; otherwise it is *oblique*. [96]

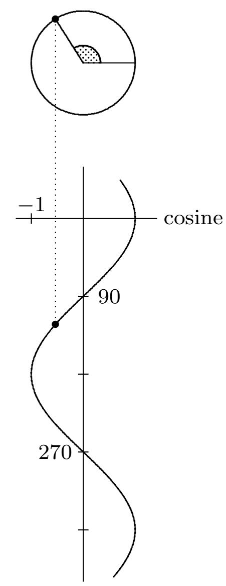
confocal conics: Ellipses or hyperbolas that share focal points. [62] When an ellipse and a hyperbola are confocal, they intersect perpendicularly. [596, 847]

conic section: Any graph obtainable by slicing a cone with a cutting plane. This might be an ellipse, a parabola, a hyperbola, or some other special case.

conversions: 1 mile = 5280 feet; 1 foot = 12 inches; 1 inch = 2.54 centimeters; one liter is 1000 milliliters; a milliliter is the same as a cubic centimeter.

cosecant: Usually abbreviated csc, this is the reciprocal of the sine ratio. See *secant* for a diagram. This function makes it convenient to express some trigonometric results without writing an explicit division — for example, $\dfrac{12.8}{\sin 25}$ is the same as $12.8 \csc 25$. Do *not* confuse cosecant with $\sin^{-1}$. [621, 779]

cosine: This is a combination of *complement* and *sine*, so named because the cosine of an angle is the same as the sine of the complementary angle.



cosine graph: The standard polar angle determines a point on the unit circle. The graph shown at right relates the standard angle to the projection of this point onto the *horizontal* axis. Because it is customary to graph functions with the domain variable (the angle, in this case) plotted horizontally, this is not the usual presentation. [183]

cosines of supplementary angles: The cosine of an angle is the *opposite* of the cosine of the supplement: $\cos\theta = -\cos(180-\theta)$ or (in radian mode) $\cos\theta = -\cos(\pi-\theta)$ [153, 184, 257]

cotangent: The reciprocal of the tangent. As the name suggests, it is *also* the complementary function: $\cot\theta = \tan(90-\theta)$ or (in radian mode) $\cot\theta = \tan\left(\frac{1}{2}\pi - \theta\right)$ [819]

cyclic: A polygon, all of whose vertices lie on the same circle, is called *cyclic*.

cycloid: A curve traced by a point on a wheel that rolls without slipping. [131, 174, 279, 463]

decibel: A unit used when comparing the power of two acoustic signals. The loudness of a sound whose intensity is I is said to be $10 \cdot \log(I \div I_0)$ *decibels*, where I_0 is the intensity of a barely audible sound. This is a meaningful scale, for there is evidence that the intensity of a sound (the rate at which energy bombards the eardrum) is related logarithmically to the perceived loudness of the sound. This illustrates how effective logarithms are when it is necessary to deal simultaneously with very large and very small numbers. [700]

determinant: A ratio that is associated with any square matrix. Except for a possible sign, the determinant of a 2×2 matrix $\mathbf{M}$ is the area of any region $\mathcal{R}$ in the xy plane, divided into the area of the region that results when $\mathbf{M}$ is applied to $\mathcal{R}$. [391]

dihedral: An angle that is formed by two intersecting *planes*. [36, 552]

dilation: A similarity transformation that has a fixed point C (the center of the dilation) and that moves all other points towards C or away from C; there is a magnification factor k so that the image of every point P is $C + k \cdot \overrightarrow{CP}$. [37, 481]

directrix: *Ellipses* and *hyperbolas* have two such lines; a *parabola* has only one. [493, 567, 741]

domain: The domain of a function consists of all numbers for which the function gives a value. For example, the domain of a logarithm function consists of positive numbers only. [363, 716]

dot product: The dot product of two vectors $\mathbf{u} = [a, b]$ and $\mathbf{v} = [m, n]$ is the number $\mathbf{u} \bullet \mathbf{v} = am + bn$. The dot product of two vectors $\mathbf{u} = [a, b, c]$ and $\mathbf{v} = [p, q, r]$ is the number $\mathbf{u} \bullet \mathbf{v} = ap + bq + cr$. In either case, it is *the sum of the products of corresponding components.* Because the product of two vectors is a *number*, this operation is sometimes called the *scalar product.* It has the familiar commutative property $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$ and distributive property $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$. [101]

doubling time: When a quantity is described by a increasing *exponential function of t*, this is the time needed for the current amount to double. [417]

eccentricity: For curves defined by a focus and a directrix, this nonnegative number determines the shape of the curve. It is the distance to the focus divided by the distance to the directrix, measured from any point on the curve. The eccentricity of an ellipse is less than 1, the eccentricity of a parabola is 1, and the eccentricity of a hyperbola is greater than 1. The eccentricity of a circle (a special ellipse) is 0. The word is pronounced "eck-sen-trissity". [481, 499, 741]

ellipse I: An ellipse has two focal points. The sum of the *focal radii* to any point on the ellipse is constant. [430,454]

ellipse II: An ellipse is determined by a focal point, a directing line, and an eccentricity between 0 and 1. Measured from any point on the curve, the distance to the focus divided by the distance to the directrix is always equal to the eccentricity. [493]

excenter If two sides of a triangle are extended, the center of the circle that is tangent to the two extended sides as well as the third side of the triangle is called an excenter. Each triangle has three excenters. [79]

explicit function An explicit definition for a function is one containing no reference to the function itself. See *recursive*. [637]

exponential functions are defined by $f(x) = a \cdot b^x$, with a positive constant base and a variable exponent. Do not confuse these with power functions! [400,492,534]

exponents, rules of: These apply when there is a *common base*: $a^m \cdot a^n = a^{m+n}$ and $\dfrac{a^m}{a^n} = a^{m-n}$; when there is a *common exponent*: $a^m \cdot b^m = (a \cdot b)^m$ and $\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$; or when an exponential expression is raised to a power: $(a^m)^n = a^{mn}$. In particular, $a^0 = 1$ and $a^{-m} = \dfrac{1}{a^m}$ and $a^{1/m} = \sqrt[m]{a}$. [292, 315, 342, 353, 368, 375]

extrapolate: To enlarge a table of values by going outside the given range of data. [395]

factorial: The product of all positive integers less than or equal to n is called n *factorial*. The abbreviation $n!$ is generally used. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. [834]

Ferris wheel: George Ferris built the first one for the 1893 Columbian Exposition in Chicago, and it held the record for size until 2000. Supported by two 140-foot towers, a wheel 250 feet in diameter rotated about a horizontal axis, carrying 36 compartments, each of which was designed to hold 60 passengers. [115, 162, 362, 658]

Fibonacci: The nickname of Leonardo of Pisa (ca. 1180-1250), an Italian merchant who traded with north Africa and helped transmit the mathematical knowledge and notation of the Arab world to Europe. He wrote an algebra book, which included a famous problem about rabbits, the answer to which is now known as the Fibonacci sequence. [568]

focus: Ellipses and hyperbolas have two such points; a *parabola* has only one. [454, 567, 766]

focal radius: A segment that joins a point on a conic section to one of the focal points; also used to indicate the length of such a segment. [442, 454, 766]

fractal: A non-trivial configuration of points that typically presents the same pattern, no matter how much it is magnified. [548]

frustrum: There is no such word. See *frustum*.

frustum: When a cone or pyramid is sliced by a plane that is parallel to its base, one of the two pieces is another cone or pyramid; the other is a frustum. [37, 272]

function: A function f is a rule that assigns an unambiguous output value $f(v)$ to each input value v in its *domain*.

future value: The expected value of an investment after a specified period of growth at a fixed rate of interest. [693]

geometric mean: If x and y are positive numbers, $\sqrt{xy}$ is their *geometric mean*. [711]

geometric sequence: A list in which each term is obtained by applying a constant multiplier to the preceding term. [547]

geometric series: The sum of a geometric sequence. To evaluate such a sum, you only need to know the first term a, the last term b, and the multiplier r; the sum is $\dfrac{a-br}{1-r}$. [682]

great circle: If a sphere is sliced by a plane, the intersection curve is a circle. If the plane goes through the center of the sphere, the circle is called a *great circle*. The earth's *equator* is a great circle, and so are the *longitude* lines (also called *meridians*) that run from pole to pole. With the exception of the equator, lines of latitude are *not* great circles. [260]

Greek letters appear often in mathematics. Some common ones are α (alpha), β (beta), Δ or δ (delta), θ (theta), Λ or λ (lambda), μ (mu), π (pi), Σ (sigma), and Ω or ω (omega).

half-life: When a quantity is described by a *decreasing exponential function of* t, this is the time needed for half of the current amount to disappear. [408, 757]

half-turn: Descriptive name for a 180-degree rotation. [43]

head: Vector terminology for the second vertex of a directed segment.

Heron's formula: The area of a triangle can be calculated from SSS information by the formula $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$ is half the perimeter of the triangle. [7305]

hyperbola I: A hyperbola has two focal points, and the difference between the *focal radii* drawn to any point on the hyperbola is constant. [715, 766]

hyperbola II: A hyperbola is determined by a focal point, a directing line, and an eccentricity greater than 1. Measured from any point on the curve, the distance to the focus divided by the distance to the directrix is always equal to the eccentricity. [741]

identity matrix: A square matrix in which every entry on the main diagonal — which runs from the upper left corner to the lower right corner — is 1, and all others are 0. [233]

incenter: The angle bisectors of a triangle are concurrent at this point, which is equidistant from the sides of the triangle.

initial ray: See *standard position.*

inscribed angle: An angle formed when two chords meet at a point on the circle. An inscribed angle is *half* the angular size of the arc it intercepts. In particular, an angle that intercepts a semicircle is a *right* angle.

inscribed polygon: A polygon whose vertices all lie on the same circle; also called *cyclic.*

interpolate: To enlarge a table of values by staying within the given range of data. See also *linear interpolation.* [433]

inverse function: Any function f processes input values to obtain output values. A function that reverses this process is said to be *inverse* to f, and is often denoted f^{-1}. In other words, $f(a) = b$ must hold whenever $f^{-1}(b) = a$ does. For some functions ($f(x) = x^2$, for example), it is necessary to restrict the domain in order to define an inverse. Notice that f^{-1} does *not* mean reciprocal. [585, 632, 697, 828]

inverse matrix: Given a square matrix $\mathbf{M}$, the inverse matrix (if such exists) is the unique square matrix $\mathbf{M}^{-1}$ for which $\mathbf{M}\mathbf{M}^{-1}$ is the *identity matrix.* There are square matrices that do not have inverses. [229, 425]

Inverse-Square Law: This natural law describes the rate at which radiant energy weakens as the distance from its source increases. [414]

IOKA: The movie house in downtown Exeter, from 1915 to 2008. [144]

Kepler's First Law: Planets travel in elliptical orbits, with the sun at one focus. [430]

Kepler's Third Law: Divide the cube of the mean distance from a planet to the sun by the square of the time it takes for the planet to complete its orbit around the sun — the result is the same number k for every planet. The ratio depends only on the units used in the calculation. In other words, $d^3 = kt^2$. If distances are expressed in astronomical units and time in Earth years, k equals 1. The theory applies equally well to the satellites of a planet. [433, 582, 651, 660]

Koch snowflake: Invented in 1904 by Helge von Koch, this fractal curve is the limit of an infinite sequence of polygons. [548, 786]

lateral area of a cone: For a right circular cone, the lateral area is π times the product of the base radius and the slant height, because the cone can be flattened out into a sector if it is first cut along a line from base to vertex. [98, 137, 210, 165]

lateral face: One of the faces adjacent to the base of a pyramid or prism. [2]

latitude: Given a point P on the earth's surface, its latitude is the size of the angle EOP, where O is the center of the earth, and E is the point on the equator closest to P. [187]

latitude lines: These are *circles* obtained by slicing the earth with a plane that is parallel to the equatorial plane. Their centers are on the axis that runs from pole to pole. [187, 267]

Law of Cosines: This theorem can be expressed in the SAS form $c^2 = a^2 + b^2 - 2ab\cos C$ or in the equivalent SSS form $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$. [109, 138, 146]

Law of Cosines (vector form): When two vectors $\mathbf{u}$ and $\mathbf{v}$ are placed *tail-to-tail*, the angle θ they form can be calculated by using the dot-product formula $\cos\theta = \dfrac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$. Notice the special cases: If $\mathbf{u}\bullet\mathbf{v} = 0$ then $\mathbf{u}$ is perpendicular to $\mathbf{v}$. If $\mathbf{u}\bullet\mathbf{v} < 0$ then $\mathbf{u}$ and $\mathbf{v}$ form an obtuse angle. Thus the range of values of the function $\cos^{-1}$ matches perfectly the convention that the angle formed by two vectors is no larger than a straight angle. [164]

Law of Refraction: See *Snell's Law*.

Law of Sines: This theorem says that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ holds for any triangle, where a, b, and c are the lengths of the sides opposite angles A, B, and C, respectively. All three ratios are equal to the diameter of the circumscribed circle. [135, 150]

least squares: A method of minimizing residuals when fitting a line to a scatter plot. It takes its name from the Pythagorean distance formula. [333, 344]

light-year: Approximately 5.88 trillion miles, this is a unit of length used in astronomical calculations. As the name implies, it is the distance traveled by light during one year. [294]

linear interpolation: To calculate coordinates for an unknown point that is between two known points, this method makes the assumption that the three points are collinear.

linear relationship: If $au + bv = c$, where a, b, and c are constants, then u and v are said to be *linearly related* variables. [545]

logarithm: Another name for an exponent; specifically, the exponent needed to express a given positive number as a power of a given positive base. Using 4 as the base, the logarithm of 64 is 3, because $64 = 4^3$. [470, 478, 489]

logarithms, rules of: These are exponential rules in disguise, because logarithms are exponents: $\log ab = \log a + \log b$ and $\log \dfrac{a}{b} = \log a - \log b$ and $\log\left(a^k\right) = k \log a$ hold for any base, any positive numbers a and b, and any number k; the change-of-base formula $\log_c a = \dfrac{\log a}{\log c}$ holds for any base, and any positive numbers a and c. [473, 489, 496, 516, 517]

logistic equation: A refinement of the exponential-growth model; instead of assuming that the birth rate and the death rate are constant, it is assumed that they each depend linearly on the size of the population. [726, 797]

longitude line: Great semicircle that runs from pole to pole; a *meridian*. [277, 359, 413]

magnitude: The magnitude of a vector $\mathbf{u}$ is its length, denoted by the absolute-value signs $|\mathbf{u}|$. Sometimes the notation $\|\mathbf{u}\|$ is used. [128]

major/minor arc: Of the two arcs determined by a given chord, the smaller one is called *minor*, and the larger one is called *major*.

major/minor axis: Ellipses and hyperbolas have two axes of reflective symmetry. The major axis contains the vertices and the focal points. [454, 766]

Markov chain: A sequence of vectors, each obtained from its predecessor by multiplying by a fixed square *transition matrix* $\mathbf{M}$, composed of *probability vectors*. [665, 666, 687]

matrix: This is a rectangular array of numbers. It is called an $m \times n$ matrix if it has m rows and n columns. A vector can be written either as a column matrix or as a row matrix. The plural of matrix is *matrices*. There is no such word as *matrice*. [143]

matrix multiplication: The product of two matrices is another matrix, calculated by forming *scalar products* of row vectors from the left matrix and column vectors from the right matrix. Each value is stored in the row and column defined by the vectors that were multiplied. For such a product to make sense, the first matrix must have just as many columns as the second matrix has rows. [145, 152, 157, 166, 168]

mean distance: In astronomical parlance, the mean distance is the arithmetic mean of the maximum and minimum distances of an orbiting celestial object to another more massive celestial object located (approximately) at a focus. [433, 582]

meridian: Great semicircle that runs from pole to pole; a *longitude line*. [277, 359, 413]

merry-go-round: Called a *carrousel* when it was invented in eighteenth-century France, this is a large, horizontal, rotating disk with a variety of places for riders to sit. [104, 562]

minute: One fiftieth of a standard class period; also one sixtieth of a degree. [50, 429]

national anthem: See *Star-Spangled Banner*.

nautical mile: Once defined to be the length of a 1-minute arc on a meridian. [429]

Newton's Law of Cooling is described by exponential equations $D = D_0 b^t$, in which t represents time, D is the difference between the temperature of the cooling object and the surrounding temperature, D_0 is the initial temperature difference, and b is a positive constant that incorporates the rate of cooling. Isaac Newton (1642-1727) contributed deep, original ideas to physics and mathematics. [742]

non-invertible: Describes a matrix whose *determinant* is zero. Such a matrix does not have a *matrix inverse*; it is also called *singular*. [426]

parabola: This curve consists of all the points that are equidistant from a given point (the *focus*) and a given line (the *directrix*). [300, 567, 64#11]

parameter: A variable constant. [451, 811]

Pascal's triangle: The entries in the n^{th} row of this array appear as coefficients in the expanded binomial $(x + y)^n$. Each entry in the array is the sum of the two entries above it. Blaise Pascal (1623-1662) was a French philosopher, mathematician, and theologian. He invented the barometer, and made original contributions to the theory of probability. [783]

period: A function f has positive number p as a period if $f(x + p) = f(x)$ holds for all values of x. The smallest such p, if there is one, is called *the period* of f. [392, 393, 737]

permutation: An arrangement of objects. There are ${}_nP_r = n(n-1)\cdots(n+1-r)$ ways to arrange r objects that are selected from a pool of n objects. [863]

phase shift: An angle that describes how a sinusoidal curve can be obtained by translating the standard sine curve along the horizontal axis. [752]

point-slope form: One way to write a linear equation, as in $y = m(x - 3.14) + 2.72$.

polar coordinates: Given a point P in the xy-plane, a pair of numbers $(r;\theta)$ can be assigned, in which r is the distance from P to the origin O, and θ is the size of an angle in standard position that has OP as its terminal ray. [218, 234]

power functions look like $f(x) = a \cdot x^p$, with a variable base and a constant exponent. Do not confuse these with exponential functions! [590]

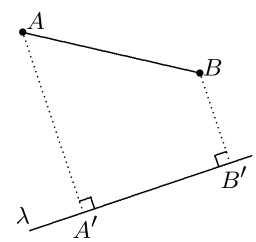
present value: The amount of money that needs to be invested so that it will grow to a certain amount (the *future value*) in a specified time, given a fixed rate of growth. [693]

prism: A three-dimensional figure that has two congruent and parallel *bases*, and parallelograms for its remaining *lateral faces*. If the lateral faces are all rectangles, the prism is a *right prism*. If the base is a regular polygon, the prism is called *regular*. [2]

prime meridian: The great semicircle that runs through Greenwich, England on its way from the North Pole to the South Pole. Points on this meridian are all said to have longitude 0 degrees. [297]

probability vectors have nonnegative components, whose sum is 1. [666,777]

projection: Given a figure $\mathcal{F}$, its perpendicular projection $\mathcal{F}'$ onto a line λ (*Greek* “lambda”) or onto a plane $\mathcal{P}$ is obtained by marking, for each point P of $\mathcal{F}$, the point P' on λ or on $\mathcal{P}$ that is closest to P. [75, 100, 192]



pyramid: A three-dimensional figure that is obtained by joining all the points of a polygonal *base* to a *vertex*. Thus all the lateral faces of a pyramid are triangles. If the base polygon is regular, and the lateral edges are all congruent, then the pyramid is called *regular*. [19, 26]

Pythagorean identity $(\cos\theta)^2 + (\sin\theta)^2 = 1$ is a consequence of the definition of the circular functions as coordinates on the unit circle. It is usually written $\cos^2\theta + \sin^2\theta = 1$, without using parentheses. [6, 45, 63]

quadratic formula: $x = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ are the two solutions to $ax^2 + bx + c = 0$.

quarter-turn: Descriptive name for a 90-degree rotation. [65]

radian: Approximately 57.3 degrees in size, this angle is defined by a circular sector whose radius and arc length are equal. [369]

radian measure: Given a circle centered at O, and one of its arcs PQ, the radian measure of angle POQ is the length of arc PQ divided by the length of radius OP. [389]

range: The range of a function consists of all possible values the function can return. For example, the range of the sine function is the interval $-1 \le y \le 1$. [717]

rational number: A number that can be expressed as the ratio of two whole numbers. [378, 732]

recursion: See *recursive description*.

recursive description: This is a method of describing a sequence, whereby one term (typically the initial term) is given and each subsequent term is defined by referring to previous terms. It is also simply referred to as *recursion*. One example: *each term is two thirds of its predecessor*. Another example: *each term is twice the predecessor minus the square of the predecessor*. [548, 637, 725, 726]

reflection property of the parabola: The line that meets a parabola tangentially at any point P on the parabola makes equal angles at the point of tangency with the line through P parallel to the axis of symmetry and the line through P and the focus. [668]

reflection property of the ellipse: A line that meets an ellipse tangentially makes equal angles with the focal radii at the point of tangency. [575, 607, 656, 794]

reflection property of the hyperbola: A line that meets a hyperbola tangentially makes equal angles with the focal radii at the point of tangency. [795, 899]

regression line: A line that has been fitted to a scatter plot. See *least squares*. [333]

relatively prime: Integers are said to be relatively prime or coprime if their only positive common divisor is 1. [486]

residual: Given a line $y = mx + b$ and a point (x_1, y_1) not on the line, the difference $y_1 - (mx_1 + b)$ is called a *residual*. Its magnitude is the vertical distance between the point and the line. Its sign tells whether the point is above or below the line. [333, 344]

rotation matrix: To rotate a column vector $\mathbf{v}$ counterclockwise through an angle θ, left-multiply it by the matrix $\mathbf{M} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. [199, 200, 219]

row vector: A single row of a matrix can be thought of as a vector. [144]

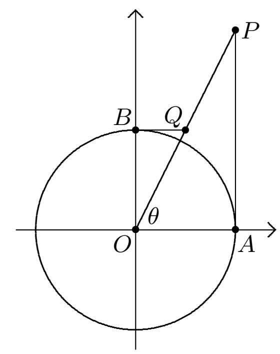
scalar: Applied to vectors, this is just another word that means *number*. For example, in the equation $\mathbf{w} = 3.2\mathbf{u} + 0.8\mathbf{v}$, the numbers 3.2 and 0.8 are scalars.

scalar product: Another name for the *dot product*.

scatter plot: The graph of a discrete set of data points.

scientific notation: The practice of expressing numbers in the form $a \times 10^n$, in which n is an integer, and a is a number whose magnitude usually satisfies $1 \le |a| < 10$. [294]

secant: The reciprocal of the cosine, or $\sec\theta = \dfrac{1}{\cos\theta}$. In the figure, in which $A = (1,0)$, $B = (0,1)$, and segments OB and BQ are perpendicular, notice that $OP = \sec\theta$, $OQ = \csc\theta$, $AP = \tan\theta$, and $BQ = \cot\theta$. Do *not* confuse $\cos^{-1}$ with secant. [621, 801]



secant line: A line that intersects a (nonlinear) graph in two places. [583]

second: One sixtieth of a *minute*. [50]

sector: This is a three-sided region formed by joining the endpoints of a circular arc to the center of the circle. The size of the central angle can be any number of degrees between 0 and 360; the other two angles of the sector are right angles. [31, 46]

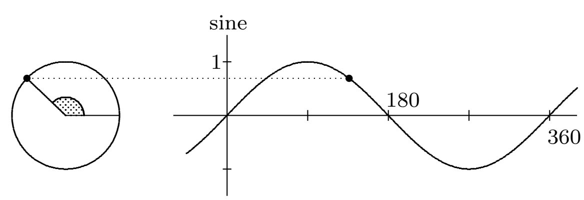
sequence: A list, typically generated according to a pattern, which can be described *explicitly*, as in $u_n = 5280(1.02)^n$, or else *recursively*, as in $u_n = 3.46u_{n-1}(1 - u_{n-1})$, $u_0 = 0.331$. [547, 874]

series: The *sum* of a sequence. [682, 703, 874]

sigma notation: A concise way of describing a *series*. For examples, the expression $\displaystyle\sum_{n=0}^{24} r^n$ stands for the sum $1+r+r^2+\cdots+r^{24}$, and the expression $\displaystyle\sum_{n=5}^{17} \frac{n}{24}$ stands for $\dfrac{5}{24}+\dfrac{6}{24}+\cdots+\dfrac{17}{24}$. The sigma is the *Greek* letter "S". [703]

simple harmonic motion: A sinusoidal function of time that models the movement of some physical objects, such as weights suspended from springs. [663]

sine graph: The standard polar angle determines a point on the unit circle. The graph shown at right relates the standard angle to the projection of this point onto the *vertical* axis. [183]



sines of supplementary angles: The sine of an angle is the same as the sine of the supplement: $\sin\theta = \sin(180-\theta)$ or (in radian mode) $\sin\theta = \sin(\pi-\theta)$ [153, 184, 257]

singular matrices: See *non-invertible*.

sinusoidal: Formally, a graph has this property if it can be described by an equation $y = a\sin(mx + b) + k$, for some numbers a, m, b, and k. The value of $|a|$ is called the *amplitude*. According to this definition, any graph $y = a\cos(mx + b) + k$ is also sinusoidal. Informally, a graph has this property if it is shaped like, and has the symmetry properties of, a sine (or cosine) curve. [302, 398, 416, 708, 721, 729, 743]

slant height: In a cone, this is the length of a segment that joins the vertex of the cone to the base; in a pyramid, it is the length of an altitude of a lateral face. [140]

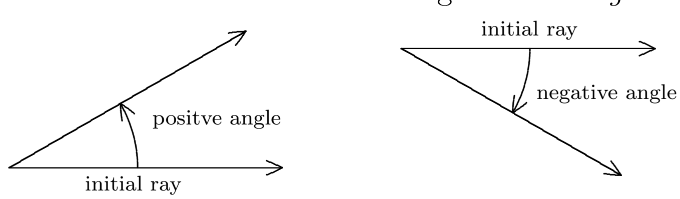
slope of a curve at a point: The slope of the tangent line at that point. [583, 630, 640, 646, 647]

snowflake: See *Koch snowflake*.

some: To a mathematician, this word means *not none*. [800]

sphere: This surface consists of all points that are at a constant distance from a *center*. The common distance is the it radius of the sphere. A segment joining the center to a point on the sphere is also called a *radius*. [169]

standard position: An angle in the xy-plane is said to be in *standard position* if its *initial ray* points in the positive x-direction. The other ray that forms the angle is referred to as the *terminal ray*. Angles that open in the counterclockwise direction are regarded as *positive*, while angles that open in the clockwise direction are regarded as *negative*. [218, 234]



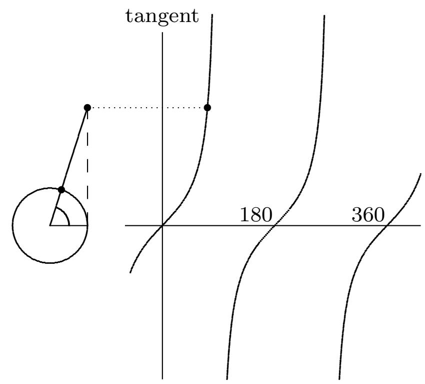
Star-Spangled Banner: See *national anthem*.

subtended angle: Given a point O and a figure $\mathcal{F}$, the angle subtended by $\mathcal{F}$ at O is the smallest angle whose vertex is O and whose interior contains $\mathcal{F}$. [32]

tail: Vector terminology for the first vertex of a directed segment.

tail-to-tail: Vector terminology for directed segments sharing a common first vertex. [18]

tangent graph: The standard polar angle determines a point on the unit circle. The graph shown at right relates the standard angle to the *slope* of the radial segment joining this point to the origin. Because vertical lines do not have slopes, the graph is disconnected whenever the angle is an odd multiple of 90 degrees. [213]



tangent to a circle: A line that touches a circle without crossing it. Such a line is perpendicular to the radius drawn to the point of tangency.

terminal ray: See *standard position*.

terminator: The circle where day meets night. [319]

tesselate: To fit non-overlapping tiles together to cover a planar region. [76, 125]

transition matrix: See *Markov chain*.

triangle inequality: The inequality $PQ \le PR + RQ$ says that any side of any triangle is at most equal to the sum of the other two sides. [198]

triangular numbers: Like square numbers, these are based on shape. [580, 790]

trigonometric addition formulas: For any angles α and β

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \text{and} \quad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

[154, 219, 239, 287, 288, 325, 749, 751, 753]

unit circle: This circle consists of all points that are 1 unit from the origin O of the xy-plane. Given a point P on this circle, the coordinates of P are the *cosine* and the *sine* of the counterclockwise angle formed by segment OP and the positive x-axis. [30]

unit square: Its vertices are $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. [188]

vectors have *magnitude* (size) and *direction*. Visualize them as directed segments (arrows). Vectors are described by *components*, just as points are described by coordinates. The vector from point A to point B is often denoted $\overrightarrow{AB}$, or abbreviated by a boldface letter such as $\mathbf{u}$, and its magnitude is often denoted $\left|\overrightarrow{AB}\right|$ or $|\mathbf{u}|$. [11]

vector projection: The vector projection of $\mathbf{v}$ onto $\mathbf{u}$ is described by $|\mathbf{v}|\,\dfrac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\,\dfrac{1}{|\mathbf{u}|}\,\mathbf{u}$, which can in turn be simplified to just $\dfrac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{u}\bullet\mathbf{u}}\,\mathbf{u}$. The vector projection of $\mathbf{v}$ onto $\mathbf{u}$ is sometimes denoted $\mathbf{proj}_{\mathbf{u}}\mathbf{v}$. [167, 211]

vertex: A point where two edges of a polygon meet, or where three or more faces of a polyhedron meet. The plural is *vertices*, but "vertice" is not a word.

volume of a cone: This is one third of the product of the base area and the height, which is the distance from the vertex to the base plane. [96]

volume of a cylinder: This is the product of the base area and the height, which is the distance between the parallel base planes. [86]

volume of a prism: This is the product of the base area and the height, which is the distance between the parallel base planes. [2]

volume of a pyramid: This is one third of the product of the base area and the height, which is the distance from the vertex to the base plane. [10, 13, 17]

volumes of similar figures: If two three-dimensional figures are similar, then the ratio of their volumes equals the *cube* of the ratio of similarity. [37, 97]

volume of a sphere: The volume enclosed by a sphere of radius r is $\frac{4}{3}\pi r^3$, which (as Archimedes showed long ago) is two thirds of the volume enclosed by the circumscribed cylinder. [253]

weighted average: A sum $p_1y_1 + p_2y_2 + p_3y_3 + \cdots + p_ny_n$ is called a *weighted average* of the numbers y_1, y_2, y_3,. . . , y_n, provided that $p_1 + p_2 + p_3 + \cdots + p_n = 1$ and each *weight* p_k is nonnegative. [193]

zero: If a number x solves an equation $f(x) = 0$, then x is called a *zero of the equation*, or a *zero of the function* f. For example, the zeros of $f(x) = 49 - x^2$ are $x = 7$ and $x = -7$.